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A robust multigrid method for Isogeometric Analysis in two dimensions using boundary correction

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Highlights

- We present the first multigrid method for IgA in 2D with convergence rates which are provably robust with respect to the spline degree.
- The method is based on a new smoother based on the mass matrix and including a boundary correction.
- Rigorous analysis of the multigrid method for a model problem, proving the robust convergence.
- The resulting method can be realized with optimal computational complexity.
- Numerical experiments demonstrate the robust behavior in practice.

Abstract

We consider geometric multigrid methods for the solution of linear systems arising from isogeometric discretizations of elliptic partial differential equations. For classical finite elements, such methods are well known to be fast solvers showing optimal convergence behavior. However, the naive application of multigrid to the isogeometric case results in significant deterioration of the convergence rates if the spline degree is increased.

Recently, a robust approximation error estimate and a corresponding inverse inequality for B-splines of maximum smoothness have been shown, both with constants independent of the spline degree. We use these results to construct multigrid solvers for discretizations of two-dimensional problems based on tensor product B-splines with maximum smoothness which exhibit robust convergence rates.

Keywords: Isogeometric Analysis; Geometric multigrid; Robustness

1. Introduction

Isogeometric Analysis (IgA), introduced by Hughes et al. [1], is an approach to the discretization of partial differential equations (PDEs) which aims to bring geometric modeling and numerical simulation closer together. The

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fundamental idea is to use spaces of B-splines or non-uniform rational B-splines (NURBS) both for the geometric description of the computational domain and as discretization spaces for the numerical solution of PDEs on such domains. As for classical finite element methods (FEM), this leads to linear systems with large, sparse matrices. A good approximation of the solution of the PDE requires sufficient refinement, which causes both the dimension and the condition number of the stiffness matrix to grow. At least for problems on three-dimensional domains or the space-time cylinder, the use of direct solvers does not seem feasible. The development of efficient linear solvers or preconditioners for such linear systems is therefore essential.

For classical FEM, it is well-known that hierarchical methods, like multigrid and multilevel methods, are very efficient and show optimal complexity, that is, the required number of iterations for reaching a fixed accuracy goal is independent of the grid size. In this case, the overall computational complexity of the method grows only linearly with the number of unknowns. For high-order FEM and hp-FEM using piecewise polynomials which are globally only in C^0, too, multigrid methods have been developed (see [2–4] and the references therein).

It seems natural to extend these methods to IgA, and several results in this direction can be found in the literature. Multigrid methods for IgA based on classical concepts have been considered in [5–7], and a classical multilevel method in [8]. An approach to local refinement in IgA based on a full multigrid method can be found in [9]. It has been shown early on that a standard approach to constructing geometric multigrid solvers for IgA leads to methods which are robust in the grid size [5]. However, it has been observed that the resulting convergence rates deteriorate significantly when the spline degree is increased. Even for moderate choices like splines of degree four, too many iterations are required for practical purposes.

Recently, in [10] some progress has been made by using a Richardson method preconditioned with the mass matrix as a smoother (*mass-Richardson smoother*). The idea is to carry over the concept of operator preconditioning to multigrid smoothing: here the (inverse of) the mass matrix can be understood as a Riesz isomorphism representing the standard L^2-norm, the Hilbert space where the classical multigrid convergence analysis is developed. Local Fourier analysis indicates that a multigrid method equipped with such a smoother should show convergence rates that are independent of both the grid size and the spline degree. However, numerical results indicate that the proposed method is not robust in the spline degree in practice. This is due to boundary effects, which cannot be captured by local Fourier analysis. Similar techniques have been used to construct symbol-based multigrid methods for IgA, see [11–13].

In the present paper, we take a closer look at the origin of these boundary effects and introduce a boundary correction that deals with these effects. We prove that the multigrid method equipped with a properly corrected mass-Richardson smoother converges robustly both in the grid size and the spline degree for one- and two-dimensional problems. For the proof, we make use of the results of the recent paper [14], where robust approximation error estimates and robust inverse estimates for splines of maximum smoothness have been shown. We present numerical results that illustrate the theoretical results.

The bulk of our analysis is first carried out in the one-dimensional setting. While multigrid solvers are typically not interesting in this setting from a practical point of view, the tensor product structure of the spline spaces commonly used in IgA lends itself very well to first analyzing the one-dimensional case and then extending the results into higher dimensions. In the present work, we extend the ideas from the one-dimensional to the two-dimensional case and obtain a robust and efficiently realizable smoother for that setting. The three- and higher-dimensional case is left for future work.

In the present paper, we restrict ourselves to the case of maximum smoothness C^{p-1} for a polynomial degree p, while mentioning that multigrid methods for high-order or hp-FEM can be used to treat the case of minimum smoothness C^0, see [2–4] and the references therein. (No fully robust method in p appears to be known in the FEM literature, but methods which have only mild dependence on p are available.) The development of robust multigrid methods for intermediate cases has to be left for future work.

We mention that efforts have also been made to apply domain decomposition solvers to isogeometric discretizations in order to solve them efficiently. A domain decomposition solver called IETI based on the ideas of the finite element tearing/interconnecting (FETI) method has been proposed in [15]. An analysis of overlapping Schwarz methods for isogeometric discretizations is given in [16]. A BDDC domain decomposition preconditioner for IgA has been proposed in [17], and a BDDC preconditioner with deluxe scaling in [18]. We do not view our multigrid method as being in competition with these methods; instead we consider it a candidate for an efficient local solver. In our view, large, multi-patch IgA problems should be tackled with a domain decomposition approach, and the local patch problems which arise from the decomposition should be solved with a fast solver such as our geometric multigrid method.

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