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# Isogeometric analysis using manifold-based smooth basis functions

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## Highlights

- An isogeometric analysis technique based on manifold basis functions is proposed.
- Basis functions are constructed by blending patchwise local approximants.
- High order smoothness and approximation are achieved on irregular quadrilateral meshes.
- Near optimal finite element convergence is obtained for second and fourth order PDEs.

## Abstract

We present an isogeometric analysis technique that builds on manifold-based smooth basis functions for geometric modelling and analysis. Manifold-based surface construction techniques are well known in geometric modelling and a number of variants exist. Common to most is the concept of constructing a smooth surface by blending together overlapping patches (or, charts), as in differential geometry description of manifolds. Each patch on the surface has a corresponding planar patch with a smooth one-to-one mapping onto the surface. In our implementation, manifold techniques are combined with conformal parameterisations and the partition-of-unity method for deriving smooth basis functions on unstructured quadrilateral meshes. Each vertex and its adjacent elements on the surface control mesh have a corresponding planar patch of elements. The star-shaped planar patch with congruent wedge-shaped elements is smoothly parameterised with copies of a conformally mapped unit square. The conformal maps can be easily inverted in order to compute the transition functions between the different planar patches that have an overlap on the surface. On the collection of star-shaped planar patches the partition of unity method is used for approximation. The smooth partition of unity, or blending functions, are assembled from tensor-product b-spline segments defined on a unit square. On each patch a polynomial with a prescribed degree is used as a local approximant. In order to obtain a mesh-based approximation scheme the coefficients of the local approximants are expressed in dependence of vertex coefficients. This yields a basis function for each vertex of the mesh which is smooth and non-zero over a vertex and its adjacent elements. Our numerical simulations indicate the optimal convergence of the resulting approximation scheme for Poisson problems and near optimal convergence for thin-plate and thin-shell problems discretised with structured and unstructured quadrilateral meshes.



*Keywords:* Manifolds; Isogeometric analysis; Partition of unity method; Finite elements; Unstructured meshes; Smooth basis functions

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## 1. Introduction

The interoperability limitation of Computer Aided Design (CAD) and Finite Element Analysis (FEA) systems has become one of the major bottlenecks in simulation-based design. CAD and FEA are inherently incompatible because they use, for historical reasons, different mathematical representations. As advocated in isogeometric analysis the use of identical basis functions for CAD and FEA can facilitate their integration. Today most of the research on isogeometric analysis focuses on NURBS [1,2] and the related T-splines [3] and subdivision basis functions [4]. The inherent tensor-product structure of NURBS means that additional techniques are required for geometries that are composed out of several NURBS patches. Specifically, around extraordinary (or irregular) points where the number of patches that join together is different than four, i.e. $\upsilon \neq 4$, alternative techniques are necessary to maintain smoothness. One prevalent approach in geometric design is to introduce additional higher order patches around the extraordinary point and to ensure that all patches match up $G^k$ continuously at their boundaries. $G^k$ refers to the notion of geometric continuity and, for instance, $G^1$ implying tangent plane continuity. As first pointed out by Groisser et al. [5] and later by Kapl et al. [6], in isogeometric analysis $G^k$ leads to $C^k$ continuity because the geometry and field variables are interpolated with the same basis functions. The utility of $G^k$ constructions in isogeometric analysis with NURBS has recently been investigated in a number of papers [7–10]. $G^k$ constructions have also been explored in the context of isogeometric analysis with T-splines [11]. A different approach for dealing with extraordinary points is provided by subdivision surfaces. The neighbourhood of the extraordinary point is replaced by a sequence of nested $C^k$ continuous patches which join $C^1$ continuously at the point itself [12,13]. Subdivision basis functions for finite element analysis have originally been proposed in [14] and have been more intensely studied in a number of recent papers [15–17]. The $G^k$ constructions known from geometric design and subdivision basis functions usually do not lead to optimally convergent finite elements. The development of $G^k$ constructions that yield optimal convergence rates is currently an active area of research [6,9,10].

We introduce in this paper an isogeometric analysis technique that builds on manifold-based basis functions for geometric modelling and analysis. As known from differential geometry, manifolds provide a rigorous framework for describing and analysing surfaces with arbitrary topology, see [18,19]. Manifold techniques for mesh-based construction of smooth $C^k$ continuous surfaces were first introduced by Grimm et al. [20]. Other mesh-based manifold constructions have later been proposed, e.g., in [21–24]. In all these approaches, a manifold surface in Euclidean space $\mathbb{R}^3$ is obtained by mapping and blending together planar patches from $\mathbb{R}^2$. In the resulting approximation scheme, similar to splines, a $C^k$ continuous surface is described with a quadrilateral or triangular control mesh and each vertex has a corresponding basis function with a local support, see Fig. 1. In contrast to the aforementioned $G^k$ constructions, which rely on matching up separate patches, in the manifold-based technique considered in this paper a $C^k$ continuous surface is created by smoothly blending of overlapping patches. The idea of blending surfaces from overlapping patches is a common theme in geometric modelling and has been used, for instance, for increasing the smoothness of subdivision surfaces around the extraordinary vertices [25–27] or (meshfree) point-based surface processing [28, 29]. In Millán et al. [30,31] point-based surface blending techniques have been used for meshfree thin-shell analysis. There are also mesh-based surface constructions that use manifold techniques, but do not rely on smooth blending of patches, see, e.g., [32,33].

In the present work we follow Ying and Zorin [22] and construct smooth basis functions by combining manifold techniques with conformal parameterisations and the partition of unity method. The control mesh consists of quadrilateral elements with some extraordinary vertices (i.e. $\upsilon \neq 4$ for some non-boundary vertices) and the construction gives one basis function for each vertex. The first step is to assign each vertex of the control mesh and its adjacent elements a planar sub-mesh with the same connectivity. The sub-meshes serve as control meshes for planar surface patches, which can be understood as parameter spaces for basis functions. For $C^k$ continuous basis functions the planar patches have to have a $C^k$ smooth parameterisation. Although other choices are conceivable, the patches are parameterised using conformal (angle-preserving) maps. Since each surface point is represented on several patches, transition functions composed of conformal maps are used to navigate between adjacent patches. In the second step of the construction, on each planar patch the conventional partition of unity method (PUM) of Melenk et al. [34, 35] is used for constructing basis functions. According to PUM, the basis functions are the product of a partition of unity function and a patch specific polynomial approximant. In computer graphics literature the partition of unity function and the patch specific polynomial basis are usually referred to as the blending function and the embedding function, respectively. We use as blending functions b-splines that have zero value and $k$ zero derivatives at the patch

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