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Optimal-order isogeometric collocation at Galerkin superconvergent points

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Highlights

- We propose an optimally convergent isogeometric collocation scheme (odd degrees only).
- The proposed collocation points are a subset of the Galerkin superconvergent ones.
- Robustness and comparison with other collocation schemes assessed numerically.
- Convergence proof not yet available and numerical evidence not yet conclusive.

Abstract

In this paper we investigate numerically the order of convergence of an isogeometric collocation method that builds upon the least-squares collocation method presented in Anitescu et al. (2015) and the variational collocation method presented in Gomez and De Lorenzis (2016). The focus is on smoothest B-splines/NURBS approximations, i.e, having global C^{p-1} continuity for polynomial degree p. Within the framework of Gomez and De Lorenzis (2016), we select as collocation points a subset of those considered in Anitescu et al. (2015), which are related to the Galerkin superconvergence theory. With our choice, that features local symmetry of the collocation stencil, we improve the convergence behavior with respect to Gomez and De Lorenzis (2016), achieving optimal L^2-convergence for odd degree B-splines/NURBS approximations. The same optimal order of convergence is seen in Anitescu et al. (2015), where, however a least-squares formulation is adopted. Further careful study is needed, since the robustness of the method and its mathematical foundation are still unclear.

Keywords: Isogeometric analysis; B-splines; NURBS; Collocation method; Superconvergent points

1. Introduction

The splines-based collocation method for solving differential equations has about fifty years of history. The first references are [1,2], where cubic C^2 splines are used to solve a second order two-point boundary value problem.

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In particular, in order to achieve optimal convergence, [2] collocates a modified equation, where the modification is obtained by constructing a suitable interpolant of the true solution. An extension of this approach to multivariate (tensor-product) splines and partial differential equations is studied in [3], while extensions to m-order differential equations are found in [4] and in particular in [5], where the optimality of the method is achieved by collocating the original, unperturbed, equation at suitably selected collocation points, i.e, Gaussian quadrature points. The method only works for splines of continuity C^{m-1} and degree $m + k - 1$, with $k \geq m$. Splines-based collocation has been successfully applied also to integro-differential equations on curves, and to the boundary element method for planar domains (see [6] and references therein).

The interest and development of splines-based collocation methods for partial differential equations has been driven in the last decade by isogeometric analysis (see [7–21] and references therein). The motivation is computational efficiency: isogeometric collocation is more efficient than the isogeometric Galerkin method, at least for standard code implementations, see [22]. In particular, the assembly of system matrices is much faster for collocation than for Galerkin (unless one adopts recent quadrature algorithms as in [23]). On the other hand, contrary to the Galerkin method, isogeometric collocation based on maximal regularity splines has always been reported suboptimal in literature, when the error is measured in L^2 or L^∞ norm. For example, the L^2 norm of the error of the collocation method at Greville points, studied in [9] for a second-order elliptic problem, converges under h-refinement as $O(h^{p-1})$ or $O(h^p)$, when the degree p is odd or even, respectively, while the optimal interpolation error is $O(h^{p+1})$ regardless of the parity of p for a smooth solution. We remark that the previous ideas of [2,5] cannot be applied directly to the isogeometric case since [2] would require a complex modification of the equation (this approach however deserves further investigation) and [5] does not work for maximal smoothness splines, which represent the most interesting choice in this framework.

Collocating the equation at Greville points (obtaining the method to which we refer here as Collocation at Greville Points, C-GP) is a common choice since Greville points are classical interpolation points for arbitrary degree and regularity splines, well studied in literature, see e.g. [24]. There is however an alternative and interesting approach, from [25] and [21]. In particular, [21] introduces an ideal collocation scheme whose solution coincides with the solution of the Galerkin method, thus recovering optimal convergence. This scheme uses as collocation points the so-called Cauchy–Galerkin points, a well-chosen subset of the zeros of the Galerkin residual. These points are not known a-priori, and therefore [21] selects as approximated Cauchy–Galerkin points the points where, under some hypotheses (we will return on this point later on, in Section 3.3), one can prove superconvergence of the second derivatives of the Galerkin solution. Indeed, for a Poisson problem the residual is equivalent to the error on the approximation of the second derivatives. This is an idea from the previous paper [25]: if we constrain the numerical residual to be zero where the Galerkin residual is estimated to be zero up to higher order terms, then the computed numerical solution is expected to be close to the Galerkin numerical solution up to higher order terms as well. There are however two difficulties: the first and most relevant one is that also the superconvergent points are not known with enough accuracy everywhere in the computational domain; the second one is that there are more Galerkin superconvergent points than degrees-of-freedom, n_{dof}, for maximal smoothness splines (the superconvergent points are about $2n_{dof}$). Indeed, [25] proposes to compute a solution of the overdetermined linear system by a least-square approximation. This approach, which is more expensive than collocation, achieves optimal convergence for odd degrees and one-order suboptimal for even degrees. We refer to it as Least-Squares approximation at Superconvergent Points (LS-SP). Instead, [21] designs a well-posed collocation scheme by selecting only n_{dof} collocation points among those used in [25]. Roughly speaking, one superconvergent point per element is used as collocation point, i.e., one every other superconvergent point (as shall be clearer later), and therefore in this paper we denote this method as Collocation at Alternating Superconvergent Points (in short C-ASP). The L^2 convergence of C-ASP is one-order suboptimal for any degree, i.e., the L^2-error decays as $O(h^p)$ for any p, which means that the lack of accuracy in the estimated location of the superconvergent points affects the convergence behavior of the collocation method C-ASP.

However, we have an interesting and useful finding to report in this paper. In the framework of [21], we propose a new criterion for selecting the subset of superconvergent points, which features local symmetry and gives improved convergence properties compared to C-ASP. Roughly speaking, we propose to take two (symmetric) superconvergent points in every other element. This method, which we refer to as Clustered Superconvergent Points (C-CSP), features the same convergence order as the LS-SP approach, i.e., optimal convergence for odd degrees in L^2 and L^∞ norm. Thus, we finally have an optimally convergent isogeometric collocation scheme with cubic C^2 splines (see [22] for a discussion on the relevance of this case).

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