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A wavelet integral collocation method for nonlinear boundary value problems in physics

Lei Zhang, Jizeng Wang *, Xiaojing Liu, Youhe Zhou

Key Laboratory of Mechanics on Disaster and Environment in Western China, the Ministry of Education, College of Civil Engineering and Mechanics, Lanzhou University, Lanzhou 730000, China

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ABSTRACT

A high-order wavelet integral collocation method (WICM) is developed for general nonlinear boundary value problems in physics. This method is established based on Coiflet approximation of multiple integrals of interval bounded functions combined with an accurate and adjustable boundary extension technique. The convergence order of this approximation has been proven to be N as long as the Coiflet with $N-1$ vanishing moment is adopted, which can be any positive even integers. Before the conventional collocation method is applied to the general problems, the original differential equation is changed into its equivalent form by denoting derivatives of the unknown function as new functions and constructing relations between the low- and high-order derivatives. For the linear cases, error analysis has proven that the proposed WICM is order N, and condition numbers of relevant matrices are almost independent of the number of collocation points. Numerical examples of a wide range of nonlinear differential equations in physics demonstrate that accuracy of the proposed WICM is even greater than N, and most interestingly, such accuracy is independent of the order of the differential equation to be solved. Comparison to existing numerical methods further justifies the accuracy and efficiency of the proposed method.

1. Introduction

Nonlinear boundary value problems (BVPs) [1] arise from almost every scientific and engineering field, especially mechanical theory of beam and plate structures [2–4]. Commonly, the nonlinear BVPs can be written into a general form [5]

$$\mathbf{T}\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0, \quad x \in [a, b] \tag{1}$$

with corresponding boundary conditions, where $\mathbf{T}$ is a nonlinear operator and y is an unknown function of x.

Since the solutions of nonlinear BVPs are critically important for analyzing scientific and engineering problems, finding accurate and efficient methods for solving Eq. (1) has attracted considerable research attention. In the last few decades, numerous methods have been developed [6–12]: for example, Wang and Wu [7] proposed a fourth-order compact finite difference method (CFDM) to solve nonlinear 2nth-order multi-point boundary value problems; Geng [8] studied the nonlinear four-point boundary value problems by applying the reproducing kernel Hilbert space methods (RKHSM); Behroozifar [10] suggested a spectral method based on Bernstein polynomials (SMBP) for nonlinear differential equations with multi-point boundary conditions [10]; Liu et al. [6,12] proposed a wavelet Galerkin method (WGM) for studying nonlinear differential equations with Dirichlet and Neumann boundary conditions. Moreover, there are still many other solution methods including the shooting method, series method, function space method [13], homotopy analysis method [14] and high order finite difference method [15] etc. Although, there are many alternative ways to study the nonlinear BVPs [1], yet finding highly accurate solutions to general nonlinear BVPs remains a challenge.

Wavelet theory is a newly developed powerful mathematical tool that has shown its potential in the numerical analysis of differential equations [16,17]. Wavelet-based methods combined with conventional Galerkin [16,18] or collocation techniques [17,19] have been proposed for various nonlinear engineering problems. In our recent works [6,20–22], an efficient WGM has been proposed to solve the Bratu equation [6], large deformation bending of circular and rectangular plates [20,22], and large deflection and post-buckling analysis of nonlinearly elastic rods [21]. The numerical results have considerably better accuracy than many other numerical methods, and show their applicability to strong nonlinear problems. However, just like most other numerical methods for the

* Corresponding author.
E-mail address: jzwang@lzu.edu.cn (J. Wang).

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nonlinear BVPs, the convergence rate of the WGM can be affected by the highest derivatives involved in the equations [4,6,20,21].

In this study, we construct a modified wavelet approximation of n-tuple integrals of a function, whose accuracy is found higher than the approximations of its derivatives, and interestingly independent of the tuple of the integral. Based on this finding, we transform the original nonlinear BVPs into its equivalent forms by defining the derivatives of unknown functions as new functions and constructing relations between the low- and high-order derivatives. When solving the transformed equation by using conventional collocation method in terms of the proposed wavelet approximation on n-tuple integrals, we show that the accuracy order can be any positive even integer $N = 4, 6, 8, \ldots$, for the Coiflet with compact support $[3N-1]$. And this accuracy order is independent of the orders of the original differential equations. Both error analysis and nontrivial numerical examples are given for justifications.

2. Wavelet approximation of multiple integrals

The multi-resolution analysis of wavelet theory [23] states that the function space $\mathbf{L}^2(\mathbf{R})$ can be divided into a sequence of nested subspaces $\{0\}\cdots\subset V_0\subset V_1\subset\cdots\subset V_j\subset V_{j+1}\subset\cdots\subset\mathbf{L}^2(\mathbf{R})$. A set of orthogonal basis of the subspace V_j can be formed by

$$\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^jx-k), \quad k\in\mathbf{Z}, \tag{2}$$

where $\phi(x)$ is the so-called orthogonal scaling function. A function $f(x)\in\mathbf{L}^2(\mathbf{R})$ can be approximated by projecting this function from $\mathbf{L}^2(\mathbf{R})$ to V_j as

$$f(x)\approx\mathbf{P}^jf(x)=\sum_k c_{j,k}\phi_{j,k}(x) \tag{3}$$

where $c_{j,k} = \int_{-\infty}^{\infty} f(x)\phi_{j,k}(x)dx$, and integer j is the so-called resolution level. The scaling function with compact support can be constructed by using a finite number of low-pass filter coefficients p_k in terms of the relation below:

$$\phi(x)=\sum_k p_k\phi(2x-k) \tag{4}$$

in which subscript $k = 0, 1, 2, \ldots, 3N-1$ for the Coiflet-type wavelet, and $N-1$ is the number of vanishing moments of the corresponding wavelet function [24]. Such a scaling function has the unique property of shifted vanishing moments:

$$\int_{-\infty}^{\infty}(t-M_1)^k\phi(t)dt=0, \quad 1\le k<N \tag{5}$$

where $M_1\triangleq\int_{-\infty}^{\infty}x\phi(x)dx=\sum_{k\in Z}p_kk/2$ is the first-order moment of the scaling function. Based on this unique property, one has $c_{j,k}\approx 2^{-j/2}f(\frac{k+M_1}{2^j})$, such that the approximation of the function can be written as [25]

$$f(x)\approx\mathbf{P}^jf(x)\approx\tilde{\mathbf{P}}^jf(x)=\sum_{k=-\infty}^{\infty}f_{M_1+k}\phi(2^jx-k) \tag{6}$$

where $f_{M_1+k} = f(x_{M_1+k}), x_{M_1+k} = (M_1+k)/2^j$, and $f(x) = \mathbf{P}^jf(x) = \tilde{\mathbf{P}}^jf(x)$ when $f(x)$ is any polynomial with an order up to $N-1$, and $\left\|f(x)-\tilde{\mathbf{P}}^jf(x)\right\|_{\mathbf{L}^2(\mathbf{R})} = O(2^{-jN})$ as long as $f(x)\in\mathbf{L}^2(\mathbf{R})\cap\mathbf{C}^N(\mathbf{R})$ [26].

If the function $f(x)$ is defined on an interval, for instance, $[0, 1]$, then Eq. (6) can be rewritten as

$$f(x)\approx\tilde{\mathbf{P}}^jf(x)=\sum_{k=2-3N}^{2^j-1}f_{k+M_1}\phi\left(2^jx-k\right). \tag{7}$$

We define the n-tuple integral of the function $f(x)$ as [27,28]

$$f^{\int_n}(x)\triangleq\int_0^x\int_0^{\xi_n}\cdots\int_0^{\xi_2}f(\xi_1)d\xi_1d\xi_2\ldots d\xi_n. \tag{8}$$

Substituting Eq. (7) to Eq. (8) yields

$$f^{\int_n}(x)\approx f^{\int_n}_{\tilde{\mathbf{P}}^j}(x)=\sum_{k=2-3N}^{2^j-1}f_{k+M_1}\phi^{\int_n}_{j,k}(x), \tag{9}$$

where

$$\phi^{\int_n}_{j,k}(x)\triangleq\int_0^x\int_0^{\xi_n}\cdots\int_0^{\xi_2}\phi(2^j\xi_1-k)d\xi_1d\xi_2\ldots d\xi_n,$$
$$f^{\int_n}_{\tilde{\mathbf{P}}^j}(x)=\int_0^x\int_0^{\xi_n}\cdots\int_0^{\xi_2}f^{\int_n}_{\tilde{\mathbf{P}}^j}(\xi_1)d\xi_1d\xi_2\ldots d\xi_n.$$

Theorem 1. *If $f(x)\in\mathbf{L}^2(\mathbf{R})\cap\mathbf{C}^N(\mathbf{R})$, then the accuracy of approximation (9) can be estimated as*

$$\left\|f^{\int_n}(x)-f^{\int_n}_{\tilde{\mathbf{P}}^j}(x)\right\|_{\mathbf{L}^2[0,1]} \le 2^{-jN}\frac{\Omega(3N-1)}{N!(n-1)!}\left|(M_1-\tau)^N\Theta\phi(\tau)\right|\cdot\left\|x^{n-1}\right\|_{\mathbf{L}^2[0,1]} \tag{10}$$

and

$$|f^{\int_n}-f^{\int_n}_{\tilde{\mathbf{P}}^j}|\le 2^{-jN}\frac{\Omega(3N-1)}{N!(n-1)!}|(M_1-\tau)^N\Theta\varphi(\tau)x^{n-1}| \tag{11}$$

where $\Theta=\max[|f^{(N)}(x)|]$, $x\in[0,3N-1]$, $\Omega=1+(3N-2)/2^j$, and $\tau\in[0,3N-1]$.

Proof. Using Taylor expansion of $f(y)$ at the point x gives:

$$f(y)=\sum_{n=0}^{N-1}\left[\frac{f^{(n)}(x)}{n!}(y-x)^n\right]+\frac{f^{(N)}(\vartheta)}{N!}(y-x)^N \tag{12}$$

where $f^{(n)}(x)\triangleq d^nf(x)/dx^n$ and ϑ is on the segment connecting y and x. Assigning $y=(k+M_1)/2^j$, $k\in\mathbf{Z}$ into Eq. (12) gives the expansion of f_{k+M_1} as

$$f_{k+M_1}=\sum_{n=0}^{N-1}\left[\frac{f^{(n)}(x)}{n!}\left(\frac{k+M_1}{2^j}-x\right)^n\right]+\frac{f^{(N)}(\vartheta)}{N!}\left(\frac{k+M_1}{2^j}-x\right)^N.$$

Further considering the property of Coiflet expansion $\sum_{k\in\mathbf{Z}}(k+M_1-2^jx)^n\phi(2^jx-k)=0^n$ for $0\le n<N$, then for $x\in[0,1]$, Eq. (7) can be rewritten into [26]

$$\tilde{\mathbf{P}}^jf(x)=f(x)+2^{-jN}\sum_{2^j>k>2-3N}\frac{f^{(N)}(\vartheta_{j,k})}{N!}(k+M_1-2^jx)^N\phi(2^jx-k), \tag{13}$$

where $\vartheta_{j,k}$ becomes locating between $y=(k+M_1)/2^j$ and x, and $\mathrm{Supp}[\phi(x)]=[0,3N-1]$ is considered for determining the range of summation index k. The n-tuple integral of Eq. (13) can be expressed as

$$f^{\int_n}_{\tilde{\mathbf{P}}^j}(x)=f^{\int_n}(x)+2^{-jN}\sum_{2^j>k>2-3N}\int_0^x\int_0^{\xi_n}\cdots\int_0^{\xi_2}\frac{f^{(N)}(\vartheta_{j,k})}{N!}\times(k+M_1-2^j\xi_1)^N\phi(2^j\xi_1-k)d\xi_1d\xi_2\cdots d\xi_n. \tag{14}$$