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Mode Gaussian beam tracing

M.Yu. Trofimov *, A.D. Zakharenko, S.B. Kozitskiy

Il'ichev Pacific Oceanological Institute, 43 Baltiyskay str., Vladivostok, 690041, Russia

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ABSTRACT

A mode parabolic equation in the ray centered coordinates for 3D underwater sound propagation is developed. The Gaussian beam tracing in this case is constructed. The test calculations are carried out for the ASA wedge benchmark and proved an excellent agreement with the source images method in the case of cross-slope propagation. But in the cases of wave propagation at some angles to the cross-slope direction an account of mode interaction becomes necessary.

1. Introduction

The problem of sound propagation by the method of summation of Gaussian beams was considered in [1–6]. In these papers the absence of singularities appropriate to the ray methods (such as caustics) was proven and also the tuning of the method was performed. In the present paper we consider a method of mode Gaussian beams, which enables to treat 3D problems. In this study a method of Gaussian beams is amalgamated with the normal mode theory. More precisely, the equations for the mode amplitudes are reduced to the parabolic equations that are subsequently solved along the horizontal rays. This approach may be considered as the direct generalization of vertical modes + horizontal rays method of Burridge and Weinberg [7], yet it also incorporates some features of the mode parabolic equations theory. In the frame of the proposed method we do not consider the mode interaction.

The problem of sound propagation in a three-dimensional wedge is solved by the developed method. It appears that in the case of the cross-slope propagation the method gives very accurate results despite the absence of the mode interaction in our model. Moreover it is necessary to consider only the exited modes which present explicitly in the final solution. On the other hand, in the parabolic equation [8,9] we should consider the large number of modes with their interactions. Thus, many effects previously described in the terms of the mode interaction can be explained in the terms of the horizontal refraction.

In the case when the angle of the track of sound propagation to the across slope direction gradually increases it becomes necessary to account for the mode interaction. It is shown, that at the angle of 18° the mode interaction becomes significant and further increases. The mode interaction is therefore essential for the more general models of sound propagation.

The paper is organized as follows. After formulation of the problem in Section 2, we consider an adiabatic mode Helmholtz equation. Then, by using Babich method, we obtain a mode parabolic equation in the ray centered coordinates from the derived Helmholtz equation. In Section 4 we discuss certain details related to mode Gaussian beams propagation. In Section 5 the method of mode Gaussian beams is used for the numerical solution of the ASA wedge benchmark problem in the case of cross-slope wave propagation and in the cases of wave propagation at various angles to the cross-slope direction. The results are compared with the solutions obtained by the method of image sources and by adiabatic mode parabolic equations. The paper ends with a brief conclusion.

2. Basic equations and boundary conditions

We consider the propagation of time-harmonic sound in a three-dimensional waveguide

$$\Omega = \{(x, y, z) | 0 \le x \le \infty, -\infty \le y \le \infty, -H \le z \le 0\}$$

(the z-axis is directed upward), described by the acoustic Helmholtz equation

$$(\gamma P_x)_x + \left(\gamma P_y\right)_y + (\gamma P_z)_z + \gamma \kappa^2 P = 0, \tag{1}$$

where $\gamma = 1/\rho$, $\rho = \rho(x, y, z)$ is the density, κ is the wave-number. We assume the appropriate radiation conditions at

* Corresponding author.
E-mail address: trofimov@poi.dvo.ru (M.Yu. Trofimov).

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infinity in the x, y plane, the pressure-release boundary condition $P = 0$ at $z = 0$ and the rigid boundary condition $\partial P/\partial z = 0$ at $z = -H$. The parameters of the medium can be discontinuous at the nonintersecting smooth interfaces $z = h_1(x, y), \ldots, h_m(x, y)$, where the usual interface conditions

$$P_+ = P_-,$$
$$\gamma_+ \left(\frac{\partial P}{\partial z} - h_x \frac{\partial P}{\partial x} - h_y \frac{\partial P}{\partial y} \right)_+ = \gamma_- \left(\frac{\partial P}{\partial z} - h_x \frac{\partial P}{\partial x} - h_y \frac{\partial P}{\partial y} \right)_-, \tag{2}$$

are imposed. Hereafter the denotations $f(z_0, x, y)_+ = \lim_{z\downarrow z_0} f(z, x, y)$ and $f(z_0, x, y)_- = \lim_{z\uparrow z_0} f(z, x, y)$ are used. As will be shown below, it is sufficient to consider the case of $m = 1$, so we set $m = 1$ and denote h_1 by h.

We introduce a small parameter ϵ (the ratio of the typical wavelength to the typical size of medium inhomogeneities), the slow variables $X = \epsilon x$ and $Y = \epsilon y$ and the fast variables $\eta = (1/\epsilon)\Theta(X, Y)$ and $\xi = (1/\sqrt{\epsilon})\Psi(X, Y)$ and postulate the following expansions for the acoustic pressure P and the parameters κ^2, γ and h:

$$\begin{aligned} P &= P_0(X, Y, z, \eta, \xi) + \sqrt{\epsilon} P_{1/2}(X, Y, z, \eta, \xi) + \cdots, \\ \kappa^2 &= n_0^2(X, Y, z) + \epsilon \nu(X, Y, z, \xi), \\ \gamma &= \gamma_0(X, Y, z) + \epsilon \gamma_1(X, Y, z, \xi), \\ h &= h_0(X, Y) + \epsilon h_1(X, Y, \xi). \end{aligned} \tag{3}$$

To model the attenuation effects, we allow ν to be complex. More precisely, we put $\mathrm{Im}\nu = 2\mu\beta n_0^2$, where $\mu = (40\pi \log_{10} \mathrm{e})^{-1}$ and β is the attenuation in decibels per wavelength.

Following the generalized multiple-scale method [10], we replace the derivatives in Eq. (1) according to the rules

$$\frac{\partial}{\partial x} \to \epsilon \left(\frac{\partial}{\partial X} + \frac{1}{\sqrt{\epsilon}} \Psi_X \frac{\partial}{\partial \xi} + \frac{1}{\epsilon} \Theta_X \frac{\partial}{\partial \eta} \right),$$
$$\frac{\partial}{\partial y} \to \epsilon \left(\frac{\partial}{\partial Y} + \frac{1}{\sqrt{\epsilon}} \Psi_Y \frac{\partial}{\partial \xi} + \frac{1}{\epsilon} \Theta_Y \frac{\partial}{\partial \eta} \right).$$

Given the postulated expansions, the equation under consideration becomes

$$\begin{aligned} &\epsilon^2 \left(\frac{\partial}{\partial X} + \frac{1}{\sqrt{\epsilon}} \Psi_X \frac{\partial}{\partial \xi} + \frac{1}{\epsilon} \Theta_X \frac{\partial}{\partial \eta} \right) \Bigg((\gamma_0 + \epsilon\gamma_1) \\ &\quad \cdot \left(\frac{\partial}{\partial X} + \frac{1}{\sqrt{\epsilon}} \Psi_X \frac{\partial}{\partial \xi} + \frac{1}{\epsilon} \Theta_X \frac{\partial}{\partial \eta} \right) \cdot (P_0 + \epsilon P_1 + \cdots,) \Bigg) \\ &\quad + \text{the same term with the } Y\text{-derivatives} \\ &\quad + ((\gamma_0 + \epsilon\gamma_1)(P_{0z} + \epsilon P_{1z} + \cdots,))_z \\ &\quad + (\gamma_0 + \epsilon\gamma_1)(n_0^2 + \epsilon\nu)(P_0 + \epsilon P_1 + \cdots,) = 0. \end{aligned} \tag{4}$$

We now put

$$P_0 + \epsilon P_1 + \cdots = (A_0(X, Y, z, \xi) + \epsilon A_1(X, Y, z, \xi) + \cdots)\mathrm{e}^{\mathrm{i}\eta}.$$

Using the Taylor expansion, we can formulate the interface conditions at h_0 which are equivalent to interface conditions (2) up to $O(\epsilon)$:

$$(A_0 + \epsilon h_1 A_{0z} + \epsilon A_1)_+ = (\text{the same terms})_-, \tag{5}$$

$$\begin{aligned} &((\gamma_0 + \epsilon h_1 \gamma_{0z} + \epsilon\gamma_1) \\ &\qquad \times (A_{0z} + \epsilon h_1 A_{0zz} + \epsilon A_{1z} - \epsilon \mathrm{i}\Theta_X h_{0X} A_0 - \epsilon \mathrm{i}\Theta_Y h_{0Y} A_0))_+ \\ &\quad = (\text{the same terms})_-. \end{aligned} \tag{6}$$

2.1. The problem at $O(\epsilon^0)$

At $O(\epsilon^0)$ we obtain

$$(\gamma_0 A_{0z})_z + \gamma_0 n_0^2 A_0 - \gamma_0 \left((\Theta_X)^2 + (\Theta_Y)^2\right) A_0 = 0, \tag{7}$$

with the interface conditions $A_{0+} = A_{0-}$, $(\gamma_0 A_{0z})_+ = (\gamma_0 A_{0z})_-$ at $z = h_0$, and the boundary conditions $A_0 = 0$ at $z = 0$ and $A_{0z} = 0$ at $z = -H$. We seek a solution to problem (7) in the form

$$A_0 = B_j(X, Y, \xi)\phi(X, Y, z). \tag{8}$$

From Eq. (7) we obtain the following spectral problem for ϕ with the spectral parameter $k^2 = (\Theta_X)^2 + (\Theta_Y)^2$

$$\begin{aligned} &(\gamma_0 \phi_z)_z + \gamma_0 n_0^2 \phi - \gamma_0 k^2 \phi = 0, \\ &\phi(0) = 0, \qquad \phi_z = 0 \quad \text{at } z = -H, \\ &\phi_+ = \phi_-, \qquad (\gamma_0 \phi_z)_+ = (\gamma_0 \phi_z)_- \quad \text{at } z = h_0. \end{aligned} \tag{9}$$

This spectral problem, considering in the Hilbert space $L_{2,\gamma_0}[-H, 0]$ with the scalar product

$$(\phi, \psi) = \int_{-H}^{0} \gamma_0 \phi \psi \, dz, \tag{10}$$

has countably many solutions (k_j^2, ϕ_j), $j = 1, 2, \ldots$ where the eigenfunctions can be chosen as real functions. The eigenvalues k_j^2 are real and have $-\infty$ as a single accumulation point. The normalizing condition is

$$(\phi, \phi) = \int_{-H}^{0} \gamma_0 \phi^2 \, dz = 1. \tag{11}$$

2.2. The problem at $O(\epsilon^{1/2})$ and at $O(\epsilon^1)$

The solvability condition for the problem at $O(\epsilon^{1/2})$ is

$$\Theta_X \Psi_X + \Theta_Y \Psi_Y = 0, \tag{12}$$

from which we conclude that we can take $P_{1/2} = 0$.

2.3. The problem at $O(\epsilon^1)$

At $O(\epsilon^1)$, we obtain

$$\begin{aligned} &(\gamma_0 A_{1z})_z + \gamma_0 n_0^2 A_1 - \gamma_0 k_j^2 A_1 \\ &\quad = -\mathrm{i}\gamma_{0X} k_j A_0 - 2\mathrm{i}\gamma_0 k_j A_{0X} - \mathrm{i}\gamma_0 k_{jX} u_0 + \gamma_1 k_j^2 A_0 - \gamma_0 (\Psi_X)^2 A_{0\xi\xi} \\ &\qquad - \text{the same terms with } Y\text{-derivatives} \\ &\qquad - \frac{\partial}{\partial z}(\gamma_1 A_{0z}) - n_0^2 \gamma_1 A_0 - \nu\gamma_0 A_0, \end{aligned} \tag{13}$$

with the boundary conditions $A_1 = 0$ at $z = 0$, $A_{1z} = 0$ at $z = -H$, and the interface conditions at $z = h_0(X, Y)$:

$$\begin{aligned} &A_{1+} - A_{1-} = h_1(A_{0z-} - A_{0z+}), \\ &\gamma_{0+} A_{1z+} - \gamma_{0-} A_{1z-} \\ &\quad = h_1 \left(((\gamma_0 A_{0z})_z)_- - ((\gamma_0 A_{0z})_z)_+ \right) + \gamma_{1-} A_{0z-} - \gamma_{1+} A_{0z+} \\ &\qquad - \mathrm{i}k_j h_{0X} A_0 (\gamma_{0-} - \gamma_{0+}) - \mathrm{i}k_j h_{0Y} A_0 (\gamma_{0-} - \gamma_{0+}). \end{aligned} \tag{14}$$

Multiplying (13) by ϕ_j and then integrating the resulting equation twice from $-H$ to 0 by parts with the use of the corresponding interface conditions (14), we obtain the solvability condition for the problem at $O(\epsilon^1)$

$$\begin{aligned} &2\mathrm{i}(\Theta_{jX} B_{jX} + \Theta_{jY} B_{jY}) + \mathrm{i}(\Theta_{jXX} + \Theta_{jYY})B \\ &\quad + ((\Psi_X)^2 + (\Psi_Y)^2) B_{j\xi\xi} + \alpha_j B_j = 0, \end{aligned} \tag{15}$$

where $A_0 = B_j \phi_j$ and α_j is given by the following formula

$$\begin{aligned} \alpha_j = &\int_{-\infty}^{0} \gamma_0 \nu \phi_j^2 \, dz + \int_{-\infty}^{0} \gamma_1 \left(n_0^2 - k_j^2\right) \phi_j^2 \, dz - \int_{-\infty}^{0} \gamma_1 \left(\phi_{jz}\right)^2 dz \\ &+ \Bigg\{ h_1 \phi_j \left[\left((\gamma_0 \phi_{jz})_z\right)_+ - \left((\gamma_0 \phi_{jz})_z\right)_- \right] \\ &- h_1 \gamma_0^2 \left(\phi_{jz}\right)^2 \left[\left(\frac{1}{\gamma_0}\right)_+ - \left(\frac{1}{\gamma_0}\right)_- \right] \Bigg\} \Bigg|_{z=h_0}. \end{aligned}$$

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