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Research paper

Quasi-equal area subdivision algorithm for uniform points on a sphere with application to any geographical data distribution

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A R T I C L E I N F O

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A B S T R A C T

This paper describes a quasi-equal area subdivision algorithm based on equal area spherical subdivision to obtain approximated solutions to the problem of uniform distribution of points on a 2-dimensional sphere, better known as Smale's seventh problem. The algorithm provides quasi-equal area triangles, starting by splitting the Platonic solids into subsequent spherical triangles of identical areas. The main feature of the proposed algorithm is that the final adjacent triangles share common vertices that can be merged. It applies reshaping to the final triangles in order to remove obtuse triangles. The proposed algorithm is fast and efficient to generate a large number of points. Consequently, they are suitable for various applications requiring a large number of distributed points. The proposed algorithm is then applied to two geographical data distributions that are modeled by quasi-uniform distribution of weighted points.

1. Introduction

The problem of distributing N points uniformly over the surface of a sphere has been investigated for many decades (Robinson, 1961; Berman and Hanes, 1977; Mortari et al., 2011). This problem is one of the most challenging mathematical problems of the century and it is known as Smale's 7th problem (Smale, 1998). However, because of its implications in many areas of mathematics and its immediate practical applications in engineering, it has not only inspired mathematical researchers but also attracted the attention in various fields such as electrostatics, molecular structure, and crystallography (Saff and Kuijlaars, 1997). The capability of uniformly distributing points on a sphere has important theoretical consequences in old problems dating back to Thomson (1904) and Tammes problem (Tammes, 1930) and important applications such as survey sampling, optimization, dynamic modeling and information storage, and display in engineering, allowing the development of optimal algorithms (White, 2000; Mortari et al., 2011).

Various algorithms have been developed for a small number of points (Robinson, 1961; Berman and Hanes, 1977; Dragnev et al., 2002). However, most of them use optimization techniques that are not efficient for a large number of points. Other more modern algorithms, such as Chan's Quadrilateralized Spherical Cube Map (QSCM) projection (1975 Navy report, now out-of-print), extensively analyzed in the reference (O'Neill and Laubscher, 1976) and applied by Naval and NASA programs, and the algorithm by Snyder (1992), which is based on Platonic solids, are efficient and available. These methods all generate a total number of points (N) proportional to the number of faces of a Platonic solid; for instance, proportional to 6 (Cube or Hexahedron) for the QSCM. Teanby (2006) suggested an icosahedron-based method by subsequent quadrisection for evenly spaced binning data. Massey (2012) presented a method of constructing equal area triangles by repeatedly applying quadrisection to icosahedron and iterative equalization.

In Lee and Mortari (2013b) the authors introduced the main concepts developed in detail in this article. However, while Lee and Mortari (2013b) verified the proposed algorithms with Monte Carlo approach and the Smale's validation in the view of uniformity of distributing points, in the current manuscript the verification is not confined to the uniformity of distributing points, but to the subdivision method.

In view of this, the subdivision approach is considered to develop an algorithm to distribute a large number of points on the sphere. This paper is organized as follows. The first section of this paper provides the equations for the subdivision approach. Then, at the end of the original equal area subdivision algorithm the subsequent quasi-equal area final subdivision is provided. Finally, applications to geographical data are presented.

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2. Subdivision approach

2.1. Splitting a spherical triangle in two equal area spherical triangles

Consider the generic spherical triangle that is formed on the surface of the unit sphere by three great circular arcs intersecting pairwise in three vertices as shown in Fig. 1.

The area of a spherical triangle $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_C]$ is obtained by

$$S = A + B + C - \pi \tag{1}$$

where angles A, B, and C are the dihedral angles of the spherical triangle (Bronshtein et al., 2007).

Let a be the largest side angle, $\hat{\boldsymbol{v}}_C \cdot \hat{\boldsymbol{v}}_B = \cos a$. The problem to solve here is to find the point on the side a such that the two spherical triangles identified by the unit-vectors, $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_D]$ and $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_C, \hat{\boldsymbol{v}}_D]$, have identical areas. Since the splitting point, $\hat{\boldsymbol{v}}_D$, is co-planar to $\hat{\boldsymbol{v}}_C$ and $\hat{\boldsymbol{v}}_B$, it can be linearly expressed by the unit-vectors $\hat{\boldsymbol{v}}_C$ and $\hat{\boldsymbol{v}}_B$ as follows.

$$\hat{\boldsymbol{v}}_D = \frac{1}{\sin a}[\hat{\boldsymbol{v}}_C \sin z + \hat{\boldsymbol{v}}_B \sin(a - z)] \tag{2}$$

where z (see Fig. 1) is the side of the spherical triangle $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_D]$.

Now make use of x and y to denote the angles at the vertices of the spherical triangle $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_D]$. The area of the spherical triangle $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_D]$ is

$$S_1 = x + y + B - \pi = \frac{S}{2} = \frac{A + B + C - \pi}{2} \tag{3}$$

then

$$x + y = D = \frac{A + C + \pi - B}{2} \quad \text{and} \quad y = D - x \tag{4}$$

where D is not a new variable but a known quantity. Applying the law of cosines to the spherical triangle $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_D]$ gives

$$\cos y = \sin x \sin B \cos c - \cos x \cos B \tag{5}$$

Then, using the angle difference identity and Eq. (4), we obtain

$$\cos y = \cos D \cos x + \sin D \sin x = \sin x \sin B \cos c - \cos x \cos B \tag{6}$$

and

$$\tan x = \frac{\cos D + \cos B}{\sin B \cos c - \sin D} \quad \text{where } 0 < x < \frac{\pi}{2} \tag{7}$$

Finally, using the law of sines, $\sin z \sin y = \sin c \sin x$, with the spherical triangle $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_D]$

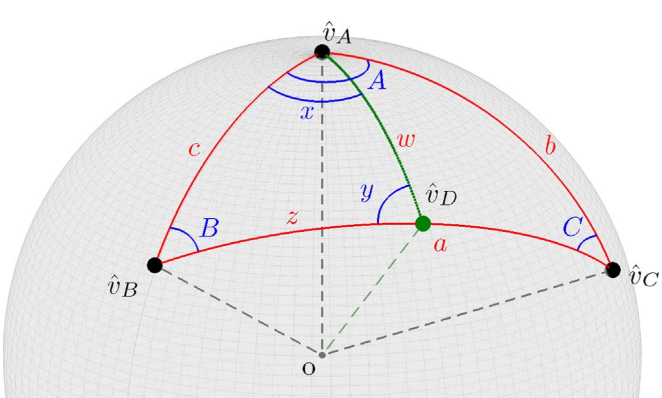

Fig. 1. Splitting a spherical triangle $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_C]$ in two equal area spherical triangles, $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_D]$ and $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_C, \hat{\boldsymbol{v}}_D]$. The angles at the vertices of the spherical triangle $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_C]$ are denoted by the upper case letters A, B, and C while the sides are denoted by lower-case letters a,b, and c. After subdivision x, y, and B are the dihedral angles and w, z, and c are the sides of the spherical triangles $[\hat{\boldsymbol{v}}_A, \hat{\boldsymbol{v}}_B, \hat{\boldsymbol{v}}_D]$.

Table 1
Platonic solids parameters. v indicates the total number of vertices, e the total number of edges, f the total number of faces, p the number of edges in each face (3 for equilateral triangles, 4 for the squares, and 5 for regular pentagons), q the number of edges meeting at each vertex. The parameter s indicates the type of initial sub-division (3 for triSection 4 for quadrisection and 5 for pentasection) to create identical triangles and i=p f is the number of initial faces.

Platonic solids	v	e	f	p	q	s	i
Tetrahedron	4	6	4	3	3	3	12
Hexahedron	8	12	6	4	3	4	24
Octahedron	6	12	8	3	4	3	24
Dodecahedron	20	30	12	5	3	5	60
Icosahedron	12	30	20	3	5	3	60

$$\sin z = \frac{\sin x \sin c}{\sin(D - x)} \tag{8}$$

is obtained and the $\hat{\boldsymbol{v}}_D$ can be computed using Eq. (2). The process can then be repeated by always splitting the longest side of the spherical triangles.

The idea of using spherical triangle splitting to generate points on a sphere finds the most natural starting point from the perfect spherical symmetry provided by Platonic solids. The parameters defining the five Platonic solids are summarized in Table 1 (Zwillinger, 2002). Since splitting a face into the number of edges with a center of face and vertices generates identical smaller triangles, initial division depends on shape of the face. Note that dual solids have same number of initial faces. Platonic solids with most initial faces are the dodecahedron and the icosahedron. For these solids the quasi-uniform distribution of points can be created by initially splitting the i=60 faces into 5 and 3 equal area triangles, respectively.

The sides of a Platonic solid can be projected onto a sphere where they form arcs. This “Platonic sphere” is the central projection of the sides of the Platonic solid onto the surface of a unit-radius sphere. The projection is on the Platonic solids' circum-sphere, which acts like a curved projection screen (Popko, 2012). All edges in Platonic solids have been transformed into geodesic arcs in corresponding platonic spheres. In platonic spheres all arcs have same length as well as all edges in Platonic solids. The vertices are corners in the case of spheres while the vertices are corner in the case of solids.

Let's show the procedure of equal-spherical area subdivision starting from an icosahedron. The vertices of an icosahedron can be defined using the Golden ratio

$$\varphi = \frac{1 + \sqrt{5}}{2} \tag{9}$$

The 12 vertices can then be obtained as all even permutations of the following set of coordinate triads

$$\left\{0,\ \pm\frac{1}{\sqrt{1 + \varphi^2}},\ \pm\frac{\varphi}{\sqrt{1 + \varphi^2}}\right\} \tag{10}$$

2.2. Equal-spherical area subdivision

For equal area subdivisions the algorithm must satisfy the following requirements:

(1) every subdivision generates triangles for recursive subdivision, and
(2) the greatest spherical dihedral angle cannot be greater than 90°. This does not allow triangles to degenerate.

It is possible to use various types of equal area subdivision which preserve area between faces in a planar triangle. However, a few