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Numerical computation of nonlinear normal modes in a modal derivative subspace

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ABSTRACT

Nonlinear normal modes offer a solid theoretical framework for interpreting a wide class of nonlinear dynamic phenomena. However, their computation for large-scale models can be time consuming, particularly when nonlinearities are distributed across the degrees of freedom. In this paper, the nonlinear normal modes of systems featuring distributed geometric nonlinearities are computed from reduced-order models comprising linear normal modes and modal derivatives. Modal derivatives stem from the differentiation of the eigenvalue problem associated with the underlying linearised vibrations and can therefore account for some of the distortions introduced by nonlinearity. The cases of the Roorda's frame model, a doubly-clamped beam, and a shallow arch discretised with planar beam finite elements are investigated. A comparison between the nonlinear normal modes computed from the full and reduced-order models highlights the capability of the reduction method to capture the essential nonlinear phenomena, including low-order modal interactions.

1. Introduction

The presence of nonlinear phenomena is often neglected in structural dynamics. However, modern engineering designs with stringent constrains on weight lead to flexible structures in which nonlinear geometric effects are significant and can no longer be neglected [1]. Nonlinearity poses important challenges as novel dynamic phenomena that cannot be treated with linear analysis can arise. An example is modal interaction where the nonlinear couplings between *linear normal modes* (LNMs) (also often referred to as vibration modes) trigger energy exchanges between modes that can potentially affect the integrity of the structure [2]. From another perspective, nonlinear dynamic phenomena can also be exploited to improve performance as, for instance, in passive micro-mechanical frequency dividers [3] and acoustic switches and rectifiers [4].

Pioneered in the 1960s by Rosenberg [5], the concept of *nonlinear normal modes* (NNMs) has proved useful to address a number of nonlinear phenomena such as mode localisation, mode bifurcation and internal resonance. NNMs can be defined as *non-necessarily synchronous periodic oscillations* of the conservative equations of motion of the system in question [6]. In the last decade, a number of numerical methods were developed to compute NNMs (see Ref. [7] for a review). Although some of these methods were successfully applied to real-life structures [8,2], the computational cost associated with the calculation of NNMs for large-scale models remains an important issue preventing the wider spread of this nonlinear analysis tool in industry. When the structure is linear with localised nonlinearities (as in Refs. [8,2]), classical linear reduction methods, such as the Craig-Bampton [9] or the Rubin-Mc Neal [10] techniques, can be used to accurately and effectively reduce the dimensionality of the linear system, leading to a significant speed-up of the NNM calculation. However, when the system possesses nonlinearities distributed among all the degrees of freedom (DOFs), as, for instance, when nonlinear geometric effects caused by finite displacements and rotations are present, such linear reduction approaches proved ineffective.

A considerable amount of research effort is directed towards the accurate prediction of the responses of nonlinear geometric systems using *reduced-order models* (ROMs). Generally, ROMs are obtained by detecting a subspace spanned by a reduced-order basis (ROB) on which the solution evolves and then by projecting the equation of motion on another basis of the same size. For structural applications characterised by symmetric jacobians, the reduction and projection bases often coincide and the method is termed

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Galerkin projection. Regardless of how the ROB is built, it is crucial to properly account for the nonlinear bending-stretching and torsion-stretching couplings triggered by the finite deflections and rotations. Reduction methods differ in the way these couplings are accounted for in the ROB. In a recent approach, termed *implicit condensation and expansion* (ICE) [11], membrane displacements are recovered from a basis of nonlinear static solutions. The computation of NNMs using this ICE method was investigated in Ref. [12]. An alternative approach to account for the membrane displacements is the concept of *modal derivatives* (MDs). Previous contributions [13–18] have shown using transient analysis that the essential distortions introduced by nonlinearity can be captured by combining LNMs and MDs in a single ROB. As it will be made clear in this paper, one of the merits of this approach lies in the systematic way MDs are derived, once the relevant LNMs have been selected for the linearised dynamics.

The transient analysis of complex nonlinear systems using, e.g., Newmark's time stepping method, is well-established. However, in addition to their substantial computational cost, time series do not bring much detailed information about the dynamics of a nonlinear system. In this context, it is therefore valuable to calculate features, such as nonlinear frequency response curves and NNMs, which give more physical insight. For instance, it can be shown that resonant vibrations, which are key to the practitioner, occur in the vicinity of nonlinear normal modes. NNMs are also able to capture modal interactions. So calculating NNMs and related frequencies is not just an intellectual exercise per se, but it is of significant engineering relevance. In this paper, the computation of NNMs of planar, geometrically nonlinear structures discretised using the finite element (FE) method is investigated. The FE models are reduced via Galerkin projection, using a ROB composed of LNMs and MDs. The accuracy of the reduction in capturing the NNMs, including modal interactions, is discussed. The paper is organised as follows. Section 2 introduces the governing equations of motion and presents the underlying theory of Galerkin projection techniques. The concept of MDs is also reviewed. The method used for computing NNMs is briefly summarised in Section 3. In Section 4, three numerical examples of geometric nonlinear models are presented: the Roorda's frame model (Section 4.1), a doubly-clamped beam (Section 4.2), and a shallow arch (Section 4.3). The NNMs obtained for the full and reduced models are compared in order to demonstrate the effectiveness of the proposed approach.

2. Governing equations and model reduction using Galerkin projection

The equations of motion of a generic free vibrating, undamped system discretised using the FE method can be written as

$$\begin{cases} \mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) + \mathbf{f}^{nl}(\mathbf{y}(t)) = \mathbf{0}, \\ \mathbf{y}(0) = \mathbf{y}_0, \\ \dot{\mathbf{y}}(0) = \dot{\mathbf{y}}_0, \end{cases} \tag{1}$$

where $\mathbf{y}(t) \in \mathbb{R}^n$ is the generalised displacement vector, $\mathbf{M} \in \mathbb{R}^{n\times n}$ and $\mathbf{K} \in \mathbb{R}^{n\times n}$ are the linear mass and stiffness matrices, respectively; $\mathbf{f}^{nl}(\mathbf{y}(t)) : \mathbb{R}^n \mapsto \mathbb{R}^n$is the nonlinear force vector. The initial conditions for the displacement and velocity vectors are denoted by $\mathbf{y}_0$ and $\dot{\mathbf{y}}_0$, respectively.

In Eq. (1), internal forces are explicitly separated in a linear and nonlinear part. In this work, the nonlinear forces $\mathbf{f}^{nl}$ emanate from geometric nonlinearities, modelled with the von-Karman kinematic assumptions that relate axial strains to the square of rotations [19]. This model yields discretised forces that are up to cubic order in $\mathbf{y}$. The von-Karman kinematic model is adequate for a large class of problems featuring elastic deflections of the order of the thickness. The numerical examples considered in this study focus on planar structures modelled with beam elements. However, the presented treatment is general and able to handle tridimensional problems, such as shells and continuum solids. From this point on, the time dependency is omitted for clarity.

In practical applications, the size n of Eq. (1) is usually large. The number of unknowns can be reduced to k, with $k \ll n$, by projecting the generalised displacement vector $\mathbf{y}$ on a suitable time-independent ROB $\mathbf{\Psi} \in \mathbb{R}^{n\times k}$ as:

$$\mathbf{y} \approx \mathbf{\Psi}\mathbf{q}, \tag{2}$$

where $\mathbf{q}(t) \in \mathbb{R}^k$ is the vector of reduced displacements. The governing equations can then be projected on the chosen basis $\mathbf{\Psi}$ in order to make the equilibrium residual orthogonal to the subspace in which the solution $\mathbf{q}$ is sought. This results in a reduced system of k nonlinear equations:

$$\mathbf{\Psi}^T\mathbf{M}\mathbf{\Psi}\ddot{\mathbf{q}} + \mathbf{\Psi}^T\mathbf{K}\mathbf{\Psi}\mathbf{q} + \mathbf{\Psi}^T\mathbf{f}^{nl}(\mathbf{\Psi}\mathbf{q}) = \mathbf{0}, \tag{3}$$

or, equivalently,

$$\hat{\mathbf{M}}\ddot{\mathbf{q}} + \hat{\mathbf{K}}\mathbf{q} + \hat{\mathbf{f}}^{nl}(\mathbf{\Psi}\mathbf{q}) = \mathbf{0}. \tag{4}$$

The reduced mass matrix $\hat{\mathbf{M}}$ and stiffness matrix $\hat{\mathbf{K}}$ do not depend on $\mathbf{q}$ and can be calculated offline. We refer to the numerical solution $\mathbf{y}$ of Eq. (1) as the *full* solution, while $\mathbf{u} = \mathbf{\Psi}\tilde{\mathbf{q}}$ is called *reduced* solution, $\tilde{\mathbf{q}}$ being the solution of Eq. (4). The key of a good reduction method is to find a suitable ROB $\mathbf{\Psi}$ that is able to accurately reproduce the full solution.

2.1. Reduction basis for geometrically nonlinear systems

The projection of the equations of motion on a basis formed with a reduced set of LNMs is a well-known approach in linear structural dynamics. Its main advantage is that the resulting ROM consists of a set of uncoupled equations that can be solved separately. For nonlinear systems, such an approach is limited because LNMs do not decouple the equations of motion and are able, by definition, to reproduce the motion only in a small neighbourhood of the equilibrium. Therefore LNMs do not constitute an effective reduction basis when the dynamics features large displacements. Some pioneering works proposed to update the ROB by computing vibration modes around arbitrary dynamic configurations attained by the system as the time integration proceeds [20–22]. However, such linear modes extracted about an arbitrary, non-equilibrium configuration do not represent the local dynamics of the motion; therefore it is not surprising that these approaches suffered from basis updates that were frequent enough to compromise the effectiveness of the method.

The ideal situation is to have a constant ROB which is able to account for the nonlinear behaviour. In that respect, MDs proved to be an effective addition to a ROB basis of LNMs, allowing the accurate transient analysis of geometrically nonlinear structures [23]. MDs stem from the directional derivatives of the linear eigenvalue problem in the direction of the LNMs. The derivation and computation of MDs are briefly reviewed for completeness.

We start by assuming a nonlinear mapping $\mathbf{\Gamma}$ between the full solution $\mathbf{y} \in \mathbb{R}^n$ and a vector of reduced linear modal coordinates $\mathbf{q} \in \mathbb{R}^k$,

$$\mathbf{y} - \mathbf{y}_{eq} = \mathbf{\Gamma}(\mathbf{q}), \tag{5}$$

where $\mathbf{y}_{eq}$ is the equilibrium configuration, and $k \ll n$. A Taylor series expansion of Eq. (5) leads to

$$\mathbf{y} - \mathbf{y}_{eq} = \left.\frac{\partial \mathbf{\Gamma}}{\partial q_i}\right|_{eq} q_i + \frac{1}{2}\left.\frac{\partial^2 \mathbf{\Gamma}}{\partial q_i \partial q_j}\right|_{eq} q_i q_j + \cdots, \tag{6}$$