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A FETI-DP based parallel hybrid stochastic finite element method for large stochastic systems

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ABSTRACT

The computational cost of uncertainty propagation in a mechanics problem can become prohibitively large as the degrees of freedom (DOF) and the number of basic random variables – also referred to as stochastic dimensionality – increase. While a number of methods have been reported in the literature to address either large DOF or high stochastic dimensionality, there is no work addressing both. This work is aimed at filling this gap. Naturally, parallel computing becomes the only feasible option for these large problems. Accordingly, a parallel domain decomposition-based hybrid method combining stochastic Galerkin and Monte Carlo simulation is developed here. To achieve scalability, which is necessary for solving very large scale problems, first the dual-primal variant of the finite element tearing and interconnecting (FETI-DP) is chosen as the domain decomposition method. Then, three distinct approaches of parallel implementation are followed. Through a set of detailed numerical experiments, scalability and relative costs of computation and communication in these three approaches are studied. Finally, based on the observations in these experiments, the best approach is selected and used to solve a large three dimensional elasticity problem with high dimensional parametric uncertainty.

1. Introduction

Modeling and propagation of uncertainties are necessary for predicting behavior of engineering systems, and they pose mathematical and numerical challenge. For instance, constitutive properties of a material [1–3], boundary conditions [4], velocity field in a fluid [5], external forces such as wind or earthquake [6,7] are modeled as random quantities. Recently, probabilistic methods are begin used for determining the material properties [8], and reliability [9] of materials with multiscale behavior. Further, parallelized open source software to study these multiscale behavior – in the context of uncertainty quantification – is proposed in [10]. A number of methods for solving stochastic mechanics problems have been proposed and studied over last few decades such as Monte Carlo simulation [11–13], perturbation [14,15], stochastic finite element method[16], to name a few. Monte Carlo simulation (MCS) is the oldest, and the most robust method for uncertainty quantification. However, computational cost becomes prohibitive for problems with large physical degrees of freedom, and sample size. To address the problem of large sample size various importance sampling scheme have been proposed [12]. Perturbation method is computationally efficient for problems with low variability. To address these computational efficiency issues SFEM was proposed, stochastic Galerkin, and stochastic collocation are in this category. Although SFEM techniques are computationally very efficient, they suffer from the *curse of dimensionality*. A detailed comparison, and analysis of various methods used for uncertainty analysis can be found in [17]. However, the computational cost can be prohibitively large in two cases. First, when the discretization of the underlying deterministic mechanics problems results in large degrees of freedom (DOF). Second, when the stochastic dimensionality – defined as the number of basic random variables – is high. In these cases, a single-processor computer becomes inadequate, and parallel computing becomes necessary. On the other hand, due to the limitation posed by the Amdahl's law [18], all methods are not amenable to efficient parallelization on large computers. For solving stochastic systems with large DOF, a number of parallel methods and implementations have been reported in the literature [19–22]. A recent trend toward this direction is development of domain decomposition (DD) based methods [23–26]. A number of methods have been proposed to address problems with large stochastic dimensionality as well, such as [27–32]. However, a parallel method for addressing both large DOF and stochastic dimensionality has not been reported in the literature. This work is aimed at filling this gap.

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The proposed parallel algorithm is based upon a DD based method developed in [33,34], specially on [34] where the finite element tearing and interconnecting (FETI) was used as the DD solver. In this method, the subdomain-level problems are solved using stochastic Galerkin and the interface problem is solved using Monte Carlo simulation. This combination results in exploiting the best features of both methods – that is, computational superiority of stochastic Galerkin and dimension independence of MCS. The novelty of the current work lies in studying the salient features of parallelization aspect of the hybrid method developed in [34]. More explicitly, the main developments in the current work over [34] are as follows. (i) The method in [34] was developed to address only large stochastic dimensionality, and accordingly, only single processor computation was performed on a system with few DOF. Whereas, in the current work both large stochastic dimensionality and DOF are considered, which demanded development of a parallel algorithm and its implementation. (ii) For achieving scalability, FETI-DP – FETI-Dual Primal – is now considered instead of FETI. (iii) The role of computer hardware is now considered for optimizing the performance. Parallel implementation of Monte Carlo is a standard practice, and has been reported in the literature [22], including around DD solvers [21]. However, the hybrid method, being a new approach that combines Monte Carlo and stochastic Galerkin, has not yet been parallelized. To the best of the authors' knowledge, this work is the first attempt in this direction.

DD methods were originally developed for solving large scale deterministic problems. For instance, FETI was proposed in [35] to solve elliptic problems with large DOF. Its implementation and performance – both numerical and parallel – in high performance computing environment are already well studied in the literature [36,37]. The numerical scalability of FETI for elasticity problems – elliptic equations in general – was numerically demonstrated in [36], and mathematically proved in [38]. However, for problems involving plates and shells – that is fourth order problems – it was observed that FETI is not numerically scalable. To achieve numerical scalability even for fourth order problems, FETI was extended to its dual primal variant FETI-DP in [39]. Development of scalable solvers using FETI and other DD methods for uncertainty quantification is in its growing stage. In [24], FETI and Krylov subspace recycling method were used to solve large stochastic systems. There FETI was used as a preconditioner and not as the main solver. A FETI based stochastic Galerkin method is proposed in [25,26,40] and detailed numerical study on numerical scalability was conducted.

Based on the results reported in [33,34], a DD approach helps in reducing the stochastic dimensionality and DOF, both in the subdomain level. On the other hand, the size of the interface problem and communication cost grows with the number of subdomains. This conflict poses a major challenge in parallelizing. To address this challenge, here we focus on parallelizing two major components of the interface problem – the iterative solver and the MCS. Three approaches are taken to this end. Parallelizing (i) the iterative solver alone, (ii) the MCS alone, and (iii) both the iterative solver and MCS. A preconditioned conjugate gradient (PCG) method is used to solve the interface problem. Scalability studies are conducted on an elliptic stochastic partial differential equation (sPDE) over a square domain. While the MCS is often termed as *embarrassingly parallel* due to its data independent nature, here it is observed that the ideal linear scalability is not achieved due to a communication bottleneck. To this end, the role of computer hardware in achieving scalability is studied. Specifically, the process-processor mapping aspect of the parallel implementation is given attention. Finally, using the results from these studies the best approach is identified and a large three dimensional elasticity problem with high stochastic dimensionality (L) is solved. Parallelization of the subdomain-level problems can be achieved by using methods such as [24,26] and is not addressed here.

The remainder of the paper is organized as follows. In the next section, (i) FETI-DP domain decomposition method, (ii) hybrid SSFEM method, and (iii) three parallel implementation approaches are discussed. Next, numerical studies on the scalability of the proposed three implementations are discussed in detail. Then, based on the numerical studies of Section 3, a large elasticity problem is solved in Section 4. Finally, concluding remarks are made and future directions are outlined.

2. The parallel hybrid SSFEM

Let the underlying probability space be denoted as $(\Omega, \mathcal{F}, P)$, with $\theta \in \Omega$ and the spatial domain be denoted as $\mathcal{D}$. $C(\boldsymbol{x}_1, \boldsymbol{x}_2)$, where $\boldsymbol{x} \in \mathcal{D}$ and $\theta \in \Omega$. The governing sPDE considered here is

$$-\nabla \cdot [\kappa(\boldsymbol{x},\theta)\nabla u(\boldsymbol{x},\theta)] = f(\boldsymbol{x},\theta) \quad a.s. \quad \text{on } \mathcal{D} \subset \mathbb{R}^d, \quad u, f \in \mathbb{R} \qquad (1)$$
$$\text{with } u(\boldsymbol{x}) = 0 \quad \text{on } \partial\mathcal{D}$$

where *a.s.* denotes *almost surely*. Spatial discretization of this equation is achieved by standard finite element discretization. The FETI-DP formulation for the resulting system of random linear equations is now discussed.

2.1. FETI-DP formulation

In FETI-DP, the domain $\mathcal{D}$ is first decomposed into N_s non-overlapping subdomains $\mathcal{D}_i$ such that $\mathcal{D} = \cup_{i=1}^{N_s} \mathcal{D}_i$. For brevity, the argument θ is omitted hereafter, unless specifically needed for clarity. Let $\boldsymbol{U}_i$ denote the discretized field variable – which is the displacement vector for elasticity problems – in the subdomain $\mathcal{D}_i$. The vector $\boldsymbol{U}_i$ is divided into corner nodes $\boldsymbol{U}_{i\mathcal{C}}$ and remaining nodes $\boldsymbol{U}_{i\mathcal{R}}$. Remaining nodes $\boldsymbol{U}_{i\mathcal{R}}$ are further divided as internal nodes $\boldsymbol{U}_{i\mathcal{I}}$ and boundary nodes $\boldsymbol{U}_{i\mathcal{B}}$. This classification of nodes is schematically shown in Fig. 1 for a two dimensional problem, with the domain decomposed into four subdomains. Using this terminology $\boldsymbol{U}_i$ can be written as,

$$\boldsymbol{U}_i = \begin{bmatrix} \boldsymbol{U}_{i\mathcal{R}} \\ \boldsymbol{U}_{i\mathcal{C}} \end{bmatrix}$$

Similarly, the stiffness matrix and the force vector in the ith subdomain can be written as

$$\boldsymbol{K_i} = \begin{bmatrix} \boldsymbol{K}_{i\mathcal{R}\mathcal{R}} & \boldsymbol{K}_{i\mathcal{R}\mathcal{C}} \\ \boldsymbol{K}^T_{i\mathcal{R}\mathcal{C}} & \boldsymbol{K}_{i\mathcal{C}\mathcal{C}} \end{bmatrix}, \quad \boldsymbol{F_i} = \begin{bmatrix} \boldsymbol{F}_{i\mathcal{R}} \\ \boldsymbol{F}_{i\mathcal{C}} \end{bmatrix},$$

where the superscript T represents matrix transpose. Compatibility of the solution across subdomains is enforced using

$$\sum_{i=1}^{N_s} \mathbf{B}_{i\mathcal{R}} \boldsymbol{U}_{i\mathcal{R}} = 0 \qquad (2)$$

where $\mathbf{B}_{i\mathcal{R}}$ are rectangular Boolean matrices with entries $1, 0$, or -1. The subdomain level vectors $\boldsymbol{U}_{i\mathcal{C}}$ can be assembled to form a vector $\boldsymbol{U}_{\mathcal{C}}$, with the mapping being

$$\boldsymbol{U}_{i\mathcal{C}} = \mathbf{B}_{i\mathcal{C}} \boldsymbol{U}_{\mathcal{C}} \qquad (3)$$

where $\mathbf{B}_{i\mathcal{C}}$ are rectangular Boolean matrices with entries 1 and 0. Using the above equations, the subdomain equilibrium equation for FETI-DP can be written as

$$\begin{aligned} \boldsymbol{K}_{i\mathcal{R}\mathcal{R}} \boldsymbol{U}_{i\mathcal{R}} + \boldsymbol{K}_{i\mathcal{R}\mathcal{C}} \boldsymbol{U}_{i\mathcal{C}} &= \boldsymbol{F}_{i\mathcal{R}} - \mathbf{B}^T_{i\mathcal{R}} \Lambda \\ \boldsymbol{K}^T_{i\mathcal{R}\mathcal{C}} \boldsymbol{U}_{i\mathcal{R}} + \boldsymbol{K}_{i\mathcal{C}\mathcal{C}} \boldsymbol{U}_{i\mathcal{C}} &= \boldsymbol{F}_{i\mathcal{C}} \end{aligned} \qquad (4)$$

$i \in 1, 2, \ldots, N_s$ and Λ is a vector of Lagrange multipliers used to enforce compatibility at boundary nodes $\mathcal{B}$. Using Eqs. (2) and (3),