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Generalization of quadratic manifolds for reduced order modeling of nonlinear structural dynamics

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ABSTRACT

In this paper, a generalization of the quadratic manifold approach for the reduction of geometrically nonlinear structural dynamics problems is presented. This generalization is obtained by a linearization of the static force with respect to the generalized coordinates, resulting in a shift of the quadratic behavior from the force to the manifold. In this framework, static derivatives emerge as natural extensions to the modal derivatives for displacement fields other than the vibration modes, such as the Krylov subspace vectors. In the nonlinear projection framework employed here, the dynamic problem is projected onto the tangent space of the quadratic manifold, allowing for a much lower number of generalized coordinates compared to linear basis methods. The potential of the quadratic manifold approach is investigated in a numerical study, where several variations of the approach are compared on different examples, giving a clear indication of where the proposed approach is applicable.

1. Introduction

With the advent of the digital era and ever so powerful computers, elaborate simulation techniques have conquered the design processes in engineering. While the computational power has increased dramatically in the past decades, the demand for more detailed and accurate models still drives the development for even more efficient and powerful simulation techniques.

In order to reduce the computational cost while preserving the accuracy of large scale models, Model Order Reduction (MOR) has found its way in the realm of nonlinear structural dynamics. Within the context of the finite element method used for the spatial discretization of arbitrary geometries, two aspects need to be tackled together in order to solve the conflicting goals of low computational effort and high accuracy. The first aspect is the reduction of the number of unknowns in the governing equations based on subspace projection. These projective MOR techniques have seen huge success, especially for linear systems, as they allow for the reduction in the number of unknowns over several orders of magnitude while retaining accuracy. The second aspect deals with the complexity reduction in the computation of certain nonlinear terms in the governing equations, which need to be updated for every evaluation in both static and dynamic problems. In this aspect, usually referred to as hyper-reduction, huge strides have been made recently in the context of finite elements [3,14].

In this paper, the first aspect, i.e. the reduction in the number of degrees of freedom (dofs) in the governing equations, is addressed. The key question of all of such projection-based techniques concerns the subspace onto which the system of equations is projected. For linear systems, various systems-theory concepts exist, such as modal decomposition, observability and controllability, transfer function or linear superposition. Many reduction techniques rely on these properties to build a suitable reduction basis, e.g. Modal Truncation, Balanced Truncation, Krylov Subspace Methods with Moment Matching or other physically-intuitive reduction techniques like the Craig-Bampton Method. An overview of the mentioned methods with application to linear structural dynamics is given in [19]. For nonlinear systems, however, these concepts either do not exist, or are not computationally feasible. For this reason, most nonlinear reduction techniques are not based on intrinsic physical properties upon which the linear methods rely, but rather are data driven, where an existing solution is analyzed in order to generate a subspace in which the solution is approximated. Despite the accuracy, all these techniques, mostly being a variant of the Proper Orthogonal Decomposition (POD), carry the drawback of the requirement of a full simulation in advance. This seems to be against the goals of MOR, especially in cases where the available computational resources do not allow

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a full simulation. The need for MOR in *simulation free* scenarios has also been a motivation for this work.

In order to build a projection subspace for nonlinear structural dynamics independent of full *a priori* simulations, some attempts have been made in the past. The key strategy is the application of an established reduction scheme on the linearized model followed by the extension of the reduction basis to capture the nonlinearity. A popular example is the use of Modal Derivatives (MDs), which are computed by means of perturbation of vibration modes (VMs) [6,1,20,7]. Furthermore, the extension of the VMs to nonlinear systems with the so called nonlinear normal modes has also been used for MOR [15], or even for demonstrating the reduction quality of the MDs [8,12].

As the computation of MDs is expensive and involves the solution of a singular system, usually a simplified *static* version of the MDs is also used (cf. companion paper [7] for a discussion). As MDs or their simplified counterparts have proved to be efficient tools for MOR [6,1,17], their concept has also been extended to other types of linear modes [5], however lacking a sound theory. A further issue in augmenting a linear basis using MDs is the quadratic growth of the basis size with respect to the number of linear modes used initially. To tackle this issue and keep the reduction basis small, selection strategies for a specific augmentation were proposed in [13] and in the companion paper [7].

In this paper, the concept of these simplified MDs is equipped with a sound theory by the use of a Quadratic Manifold (QM). The key idea lies in the mapping, not into a linear subspace (which can be interpreted as an uncurved manifold), but into a Quadratic (nonlinear) Manifold. The projection subspace is then the configuration-dependent tangent space of the manifold. As already discussed in the companion paper [7], the QM takes care of the quadratic growth issue of the linear basis size when the MDs are used as independent components.

This paper complements the companion paper [7], where the concept of QM is proposed and tested on shell structures. Here, the theory of the Static MDs is extended to displacement fields other than the VMs with a sound physical foundation. These are then called Static Derivatives (SDs). Furthermore, the proposed generalization is tested on a broader class of applications, namely 2D and 3D continuum finite element based structures.

The paper is organized as follows. In the next section, after an introduction to linear projection for MOR, the concept of nonlinear projective MOR is presented using the QM approach as an example. Then, two strategies for the computation of the necessary ingredients for the QM are discussed, namely the approach using MDs and the *Force Compensation Approach* using SDs. Thereafter, the use of SDs in the framework of a linear basis is discussed. Subsequently in Section 3, the proposed methods are applied to four examples with a focus on the accuracy of the given methods. Conclusions are given in the last section.

2. Model order reduction using quadratic manifold

2.1. Linear projective model order reduction

The semi-discretized equations of motion of a (geometrically) nonlinear structure are given as

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{g}, \tag{1}$$

where $\boldsymbol{u} \in \mathbb{R}^N$ is the vector containing the physical displacement dofs, $\boldsymbol{M} \in \mathbb{R}^{N\times N}$ is the mass matrix, $\boldsymbol{C} \in \mathbb{R}^{N\times N}$ the damping matrix, $\boldsymbol{f}(\boldsymbol{u}) \in \mathbb{R}^N$ is the nonlinear internal force vector and $\boldsymbol{g} \in \mathbb{R}^N$ is the external forcing. The time dependence of $\boldsymbol{u}$ and $\boldsymbol{g}$ is omitted for brevity. The basic idea of linear projective MOR for second order systems such as in (1) is the linear transformation of the displacement vector $\boldsymbol{u}$ to a reduced set of generalized coordinates $\boldsymbol{q} \in \mathbb{R}^n$, where $n \ll N$, such that

$$\boldsymbol{u} = \boldsymbol{V}\boldsymbol{q}, \qquad \dot{\boldsymbol{u}} = \boldsymbol{V}\dot{\boldsymbol{q}}, \qquad \ddot{\boldsymbol{u}} = \boldsymbol{V}\ddot{\boldsymbol{q}}, \tag{2}$$

where $\boldsymbol{V} \in \mathbb{R}^{N\times n}$ is the projection matrix. The columns of $\boldsymbol{V}$ span the subspace in which the high dimensional displacement vector $\boldsymbol{u}$ is constrained and the entries of $\boldsymbol{q}$ represent the time varying amplitudes of the column vectors in $\boldsymbol{V}$.

If (2) and its derivatives are substituted in (1), a residual $\boldsymbol{r}$ is expected as the system of equations are not satisfied in general. More specifically,

$$\boldsymbol{M}\boldsymbol{V}\ddot{\boldsymbol{q}} + \boldsymbol{C}\boldsymbol{V}\dot{\boldsymbol{q}} + \boldsymbol{f}(\boldsymbol{V}\boldsymbol{q}) = \boldsymbol{g} + \boldsymbol{r}. \tag{3}$$

According to the principle of virtual work, the residual force $\boldsymbol{r}$ must be orthogonal to the kinematically admissible motion $\delta\boldsymbol{u} = \boldsymbol{V}\delta\boldsymbol{q}$. In order to solve the under-determined system (3), the Galerkin projection can be used, which implies $\boldsymbol{V}^T\boldsymbol{r} = \boldsymbol{0}$. This results in the reduced system of equations

$$\boldsymbol{V}^T\boldsymbol{M}\boldsymbol{V}\ddot{\boldsymbol{q}} + \boldsymbol{V}^T\boldsymbol{C}\boldsymbol{V}\dot{\boldsymbol{q}} + \boldsymbol{V}^T\boldsymbol{f}(\boldsymbol{V}\boldsymbol{q}) = \boldsymbol{V}^T\boldsymbol{g}, \tag{4}$$

or, equivalently,

$$\boldsymbol{M}_r\ddot{\boldsymbol{q}} + \boldsymbol{C}_r\dot{\boldsymbol{q}} + \boldsymbol{f}_r(\boldsymbol{q}) = \boldsymbol{V}^T\boldsymbol{g}, \tag{5}$$

where $\boldsymbol{M}_r = \boldsymbol{V}^T\boldsymbol{M}\boldsymbol{V}$ and $\boldsymbol{C}_r = \boldsymbol{V}^T\boldsymbol{C}\boldsymbol{V}$ are the reduced linear components which can be precomputed offline, and $\boldsymbol{f}_r(\boldsymbol{q}) = \boldsymbol{V}^T\boldsymbol{f}(\boldsymbol{V}\boldsymbol{q})$ is the reduced nonlinear force. Clearly, this resulting system is of dimension $n \ll N$.

2.2. A nonlinear projective model order reduction approach: quadratic manifold

The key idea in linear projective MOR is the linear mapping between the full, high dimensional displacement vector $\boldsymbol{u}$ and the reduced variable $\boldsymbol{q}$. However, this mapping can be generalized to a nonlinear mapping $\boldsymbol{\Gamma}(\boldsymbol{z}) : \mathbb{R}^n \mapsto \mathbb{R}^N$, with $\boldsymbol{z} \in \mathbb{R}^n$ being the vector of nonlinear reduced generalized coordinates such that

$$\boldsymbol{u} = \boldsymbol{\Gamma}(\boldsymbol{z}), \qquad \dot{\boldsymbol{u}} = \frac{\partial\boldsymbol{\Gamma}}{\partial\boldsymbol{z}}\dot{\boldsymbol{z}}, \qquad \ddot{\boldsymbol{u}} = \frac{\partial\boldsymbol{\Gamma}}{\partial\boldsymbol{z}}\ddot{\boldsymbol{z}} + \frac{\partial^2\boldsymbol{\Gamma}}{\partial\boldsymbol{z}^2}\dot{\boldsymbol{z}}\dot{\boldsymbol{z}}. \tag{6}$$

The nonlinear transformation (6) is then substituted into the governing Eq. (1) and the force residual $\boldsymbol{r}$ is chosen to be orthogonal to the kinematically admissible displacements $\delta\boldsymbol{u}$ according to the principle of virtual work. For this transformation, $\delta\boldsymbol{u}$ is then given by

$$\delta\boldsymbol{u} = \frac{\partial\boldsymbol{\Gamma}}{\partial\boldsymbol{z}}\,\delta\boldsymbol{z} = \boldsymbol{P}_\Gamma\,\delta\boldsymbol{z}, \tag{7}$$

with the tangent projector $\boldsymbol{P}_\Gamma(\boldsymbol{z}) = \frac{\partial\boldsymbol{\Gamma}}{\partial\boldsymbol{z}}$ spanning the tangent subspace of the kinematically admissible displacements $\delta\boldsymbol{u}$. This results in the reduced system of equations as

$$\boldsymbol{P}_\Gamma^T\boldsymbol{M}\boldsymbol{P}_\Gamma\ddot{\boldsymbol{z}} + \boldsymbol{P}_\Gamma^T\boldsymbol{M}\frac{\partial^2\boldsymbol{\Gamma}}{\partial\boldsymbol{z}^2}\dot{\boldsymbol{z}}\dot{\boldsymbol{z}} + \boldsymbol{P}_\Gamma^T\boldsymbol{C}\boldsymbol{P}_\Gamma\dot{\boldsymbol{z}} + \boldsymbol{P}_\Gamma^T\boldsymbol{f}(\boldsymbol{\Gamma}(\boldsymbol{q})) = \boldsymbol{P}_\Gamma^T\boldsymbol{g}, \tag{8}$$

with the reduced mass matrix $\boldsymbol{P}_\Gamma^T\boldsymbol{M}\boldsymbol{P}_\Gamma \in \mathbb{R}^{n\times n}$, the reduced damping matrix $\boldsymbol{P}_\Gamma^T\boldsymbol{C}\boldsymbol{P}_\Gamma \in \mathbb{R}^{n\times n}$ and the reduced nonlinear force $\boldsymbol{P}_\Gamma^T\boldsymbol{f}(\boldsymbol{\Gamma}(\boldsymbol{z})) \in \mathbb{R}^n$. Relative to the linear projective reduced system (4), the extra term $\boldsymbol{P}_\Gamma^T\boldsymbol{M}\frac{\partial^2\boldsymbol{\Gamma}}{\partial\boldsymbol{z}^2}\dot{\boldsymbol{z}}\dot{\boldsymbol{z}}$ can be interpreted as a convective term due to the change of the basis, which is proportional to the curvature $\frac{\partial^2\boldsymbol{\Gamma}}{\partial\boldsymbol{z}^2}$ of the nonlinear transformation $\boldsymbol{\Gamma}(\boldsymbol{z})$ and to the square of the generalized velocities $\dot{\boldsymbol{z}}$.