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Improved algorithm for the detection of bifurcation points in nonlinear finite element problems

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ABSTRACT

Dealing with bifurcation points when solving large deformation finite element problems is not an easy task. Near such points, the Jacobian matrix becomes singular and the problem becomes difficult to solve numerically. In these situations, increasing heuristically the loading parameter during the simulation in order to follow the solution branch is not an option as this approach usually results in the divergence of the process. Efficient numerical techniques capable of handling the presence of bifurcation points are therefore necessary and continuation methods have proved to be powerful tools when dealing with these kind of issues. In Léger et al. (2015), a new implementation technique based on a Schur complement approach for the Moore-Penrose continuation method, which facilitates the detection of bifurcation points and enables branch following, was presented. This method has proved to perform well in most situations; however, in others (i.e. when mesh adaptation is added to the algorithm), some difficulties appear. In this paper, we therefore present an improved approach, which is much more robust, for the detection of bifurcation points in nonlinear finite element problems. Numerical examples will be presented to show the efficiency of the new approach.

1. Introduction

In recent years, the use of numerical methods to solve industrial problems is constantly increasing. With the complexity of these problems constantly increasing as well, developing robust, efficient and accurate numerical methods is a priority. This is especially true for large deformation hyperelastic problems, where convergence problems are frequently encountered. These problems can be related to different issues, but in most cases they are due to distorted elements that appear in the mesh or to the presence of bifurcation points (where the Jacobian matrix is singular). Bifurcation points, which correspond to eigenvalues of the system, are quantities of interest [26,21,27]. Detecting such points and following the different solution branches passed the critical loads associated with the bifurcation points are thus important and useful to better understand the physical properties of the problem we are solving. The analysis of structural instabilities [6,28], which includes the detection of bifurcation points, is generally based on a numerical continuation method (see [9,4,11,15,23,1,22,24]).

In a finite element context, where large systems are frequently encountered, simply looking at the sign of the determinant of the Jacobian matrix for the detection of bifurcation points in not an option as the computation of these determinants end up with machine overflows. A new implementation technique for the Moore-Penrose continuation method, which takes advantage of information already available and facilitates the detection of such points in a finite element context, was therefore introduced in [17]. To validate this new algorithm, the classical elastic beam buckling problem was considered as the bifurcation modes for this problem are well known (see Le Tallec and Vidrascu [16], Ikeda et al. [14]). The finite strain incompressible elasticity problem presented in [2,3] was also used for the validation process. Not only were we able to detect the bifurcation point presented in those papers, but we were also able to detect many more bifurcation points which will be presented in this paper. We note that, at that point, the validation process was done using fairly regular meshes.

To solve large deformation problems with the finite element method, two formulations are frequently used in practice: the total Lagrangian formulation [13] which refers to the initial configuration and the updated Lagrangian formulation [5] which refers to the most recently calculated configuration. These two formulations are mathematically equivalent, but the updated Lagrangian formulation tends to perform better in practical applications, as was shown in [18]. It was also shown in [18,19] that remeshing the initial configuration, which does not necessitate any transfer of vari-

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ables (meaning that no loss of information results from this), also enhances the performance of the updated Lagrangian method. However, when trying to incorporate the detection of bifurcation points in our complete updated Lagrangian algorithm, we noticed that remeshing had an effect on the detection of bifurcation points. Investigation led us to a better understanding of the phenomena observed and modifications were made to the bifurcation point detection algorithm in order to obtain a much more robust numerical method, which is the object of this paper.

The paper is organized as follows. Section 2 describes the implementation technique used for the Moore-Penrose continuation method and explains how this approach facilitates the detection of bifurcation points in a finite element context. Section 3 compares the results obtained when using fairly regular meshes versus adapted meshes. Section 4 describes the improved algorithm for the detection of bifurcation points while Section 5 is devoted to the validation of this improved strategy.

2. Moore-Penrose continuation method

When solving large deformation problems, a loading parameter λ, corresponding to either external forces or prescribed displacements or both, implicitly drives the deformation. This parameter is typically increased gradually until the desired loading, λ_{max}, is attained. However, increasing heuristically the loading parameter while using a standard Newton method to solve the nonlinear system of equations is highly inefficient as this approach will usually result in the divergence of the algorithm in the neighbourhood of limit points or bifurcation points. A more efficient approach is to use a continuation method where the loading parameter λ is explicitly introduced in the system of nonlinear equations, which can be expressed as:

$$F(x) = F(u, \lambda) = 0 \tag{1}$$

with $F : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$ a smooth function. The vector of unknowns, x, therefore consists of the N degrees of freedom denoted as u plus the loading parameter.

The Moore-Penrose continuation method, which was shown to be very robust in the case of large deformation problems (see [18]), is a predictor-corrector method and can be summarized as follows (see [10,17]). Starting from a known point $x^{(i)} \in \mathbb{R}^{N+1}$ on the solution curve, and given a tangent vector $v^{(i)}$ at that point, the next point $x^{(i+1)} \in \mathbb{R}^{N+1}$ on the solution curve as well as the tangent vector at that point $v^{(i+1)}$ can be obtained using the following algorithm:

- $X^0 = x^{(i)} + h v^{(i)}$
- $V^0 = v^{(i)}$
- For $k = 0, 1, 2, \ldots, k_{max}$
  1. Solve the linear system:
  $$\begin{cases} A(X^k)\,\delta_x^k = F(X^k) \\ {V^k}^\top \delta_x^k = 0 \end{cases} \tag{2}$$
  2. Solve the linear system:
  $$\begin{cases} A(X^k)T^k = A(X^k)V^k \\ {V^k}^\top T^k = 0 \end{cases} \tag{3}$$
  3. Update the solution vector:
  $$X^{k+1} = X^k - \delta_x^k$$
  4. Update the tangent vector:
  $$Z = V^k - T^k$$
  5. Normalization of the tangent vector:
  $$V^{k+1} = \frac{Z}{\|Z\|}$$
  6. If $\|F(X^k)\| \leqslant \varepsilon_F$ and $\|X^{k+1} - X^k\| \leqslant \varepsilon_x$, convergence attained:
  $$x^{(i+1)} = X^{k+1}, \quad v^{(i+1)} = V^{k+1}$$

We note that X^0 is the prediction point obtained by making a prediction step of length h in the tangential direction, $A(X^k) = [F'_u(X^k)\ F'_\lambda(X^k)]$ is a rectangular matrix of dimension $n \times (n+1)$, $\delta_x^k = \left[\delta_u^k\ \delta_\lambda^k\right]^\top$ is a correction vector of dimension $(n+1) \times 1$, k_{max} represents the maximum number of iterations allowed while ε_F and ε_x are the desired tolerances on F and x respectively. As for T^k, it represents a correction on the tangent vector. More details can be found in [17].

As can be seen, to obtain the next point on the solution curve, two systems need to be solved at each iteration. In matrix form, these systems are given by:

$$\begin{pmatrix} F'_u(X^k) & F'_\lambda(X^k) \\ {V_u^k}^\top & V_\lambda^k \end{pmatrix} \begin{pmatrix} \delta_u^k \\ \delta_\lambda^k \end{pmatrix} = \begin{pmatrix} F(X^k) \\ 0 \end{pmatrix} \tag{4}$$

and

$$\begin{pmatrix} F'_u(X^k) & F'_\lambda(X^k) \\ {V_u^k}^\top & V_\lambda^k \end{pmatrix} \begin{pmatrix} T_u^k \\ T_\lambda^k \end{pmatrix} = \begin{pmatrix} A(X^k)V^k \\ 0 \end{pmatrix} \tag{5}$$

where the vectors V^k, T^k and δ_x^k were decomposed into their components in the u and λ directions (i.e. ${V^k}^\top = \left[V_u^k\ V_\lambda^k\right]^\top$, $T^k = \left[T_u^k\ T_\lambda^k\right]^\top$ and $\delta_x^k = \left[\delta_u^k\ \delta_\lambda^k\right]^\top$). In a finite element context, constructing and solving these two systems can be very costly. A new algorithm, based on a Schur complement approach, was therefore introduced in [17]. This new approach not only solves the systems by using as much as possible information that is already available, but it can also be incorporated numerically in a finite element code by simply adding a postcondition to the Newton method solver. The following algorithm summarizes the implementation technique of the Moore-Penrose continuation method when using this approach.

- $X^0 = x^{(i)} + h v^{(i)}$
- $V^0 = v^{(i)}$
- For $k = 0, 1, 2, \ldots, k_{max}$
  1. (a) Solve to obtain the standard Newton correction $\Delta_{u,\lambda}^k$ (for a fixed value of λ):
  $$F'_u(X^k)\,\Delta_{u,\lambda}^k = F(X^k)$$
  (b) Solve for W_u^k:
  $$F'_u(X^k)\,W_u^k = F'_\lambda(X^k)$$
  (c) Calculate the Schur complement β^k:
  $$\beta^k = V_\lambda^k - V_u^k \cdot W_u^k$$
  (d) Calculate the Moore-Penrose correction for the load parameter λ:
  $$\delta_\lambda^k = -\frac{1}{\beta^k} V_u^k \cdot \Delta_{u,\lambda}^k$$
  (e) Apply the Moore-Penrose correction on the standard Newton correction: