Computers and Structures 182 (2017) 104–118

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Reformulation and extension of the thrust network analysis

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ARTICLE INFO

Article history:
Received 15 July 2016
Accepted 29 November 2016

Keywords:
Thrust network analysis
Masonry vaults
Funicular polygon

ABSTRACT

We address the Thrust Network Analysis (TNA), i.e. the methodology for modeling masonry vaults as a discrete network of forces in equilibrium with gravitational loads, first contributed by O'Dwyer and fully developed by Block and coworkers. Reducing the bias by the quoted authors in favor of a graphical interpretation of the method, we reformulate the original version of the TNA by discarding the dual grid and focusing only on the primal grid, thus significantly enhancing the computational performances. The proposed reformulation of the TNA is also extended by including horizontal forces in the analysis as well as holes or free edges in the vault. Furthermore, the coefficient matrices entering the solution scheme are obtained by assembling the separate contribution of each branch, thus avoiding the *ad hoc* node numbering and branch orientation required by alternative implementations. Numerical examples, some of which referred to vaults having a particularly complex geometry, show the effectiveness and robustness of the proposed approach in assessing the safety conditions of existing masonry vaults or in designing new ones.

1. Introduction

Stability of masonry arches and vaults is due to their shape and self-weight magnitude and distribution [1]. Stresses in masonry arches or vaults at failure are usually considerably lower then those required to cause material failure [2]. Thus, analyses assuming that material failure is critical for safety are inappropriate for this kind of structures.

This peculiarity was an advantage prior to the development of structural analysis since stability of a full scale structure could be assessed on a scale model: the real structure was erected by scaling up the dimensions of a prototype or an existing structure while keeping constant the relative proportions.

One of the first rational approaches to the stability of masonry constructions was found in the analogy between the shape of masonry arches in equilibrium and that of hanging cables in tension. Such an analogy (or catenary principle) is known since the 17th century and was first presented by Robert Hooke in a famous anagram [3].

This approach has been used since then for assessing the safety of historical constructions such as Saint Peter's dome in the Vatican City [4] or the Sagrada Familia in Barcelona [5]. For a comprehensive review of this and alternative approaches to the statics of masonry constructions the reader is referred to [6–8], while more specific studies on the computational tools specifically developed for the analysis of masonry arches and vaults are reported in [9–13].

It was merit of Heyman [2] to realize that the centenary principle could be used in combination with the limit theorems of plasticity [14], in particular with the static one, to assess the safety of masonry structures by predicting the ultimate mechanism of arches or three-dimensional (3D) skeletal structures. The application of these formulations has been carried out by using graphical or analytical methods [15–20] usually limited to two-dimensional structural models.

Existence of an equilibrated polygon representing the transmission of thrusts between the voussoirs of a masonry arch has been proved by conceptual experiments by Jenkin [21] and Barlow [22] and recent photoelastic experiments [23,24] have shown how discrete forces are transmitted between bricks of masonry walls.

A first extension to domes and vaults has been proposed in [25] by fictitiously decomposing the structure in discrete arches in equilibrium, i.e. by looking for localized networks of forces within the structure according to what is now denominated Thrust Network Analysis (TNA). The approach by O'Dwyer [25] has been further detailed in [26], in two PhD theses by Ochsendorf [27] and Block [28], and a series of papers [29–31], some of which dedicated to exploring the closely connected topics of an algebraic approach

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http://dx.doi.org/10.1016/j.compstruc.2016.11.016
0045-7949/

to graphic statics [32,33] and form finding methods for general networks [34]. A similar concept, yet very different from the computational point of view, takes inspiration from the catenary principle and employs simulated cable nets to represent the thrust network that describes equilibrium of the vaults [35,36].

On a parallel side, starting from the eighties of the 20th century, the italian school of structural and solid mechanics has deeply investigated masonry structures by adopting the Elastic No-Tension (ENT) model and framing their analysis within the broader context of structural models with unilateral constitutive relations [7]. Due to the non-smooth nature of the ENT constitutive equations several approaches have been proposed for the solution of structural problems, namely displacement-based [37,38], stress-based [39–43] and mixed [44] approaches.

Particularly interesting is the methodology exploited in the stress-based and mixed approaches since it is based on the use of polyhedral stress potentials to generate compressive membrane stresses lumped along material surfaces or thrust surfaces. Membrane stresses are described by means of a discrete network of compressive forces, according to the ENT model, thus providing a continuum justification of the TNA and, ultimately, a unification of concepts [45].

The present paper intends to represent a further contribution to the analysis of vaulted masonry structures by specifically addressing the TNA formulation by Block and coworkers. Being mainly interested to the behavior of such structures in seismic areas, we have extended the original TNA formulation in order to include the presence of horizontal forces, an issue only marginally addressed in previous formulations.

In achieving this task, we also realized that it was computationally more convenient to simplify the original TNA formulation by somehow abandoning the role of the dual grid [46], sharply and elegantly employed in [28].

Thus, at the expense of a less intuitive description of the method, we propose a reformulation of the Thrust Network Analysis by considerably reducing the number of variables involved in the optimization process which represent the core of the TNA. Moreover, with the aim of providing an efficient tool for the analysis of existing vaults or for the design of new ones, we detail how the presence of free edges can be incorporated within the proposed TNA reformulation.

The paper is organized as follows: in Section 2 we briefly recall the concept of thrust network and obtain the equations that enforce equilibrium of its nodes. The proposed reformulation of the TNA is addressed in Section 3 with specific reference to the classical problem of vertically loaded networks. Our reformulation is then extended, in Sections 3.2 and 3.3, to the case of networks with free edges and subjected to the combined effects of horizontal and vertical loads. Finally, in Section 4, we report the results obtained for networks of increasing degree of complexity in order to show the versatility and effectiveness of our formulation.

2. Equilibrium of thrust networks

It is instructive to give a brief overview of the Thrust Network Analysis (TNA) formulated in [28] in order to properly identify the points in which our formulation differs from the original one.

Actually, we introduce some simplifications that reduce the computational complexity of the method without limiting their field of application. As a matter of fact, our computational set-up of the TNA approach does not need to construct the branch-node matrix or the dual branch-node matrix largely used in the original version of the TNA. Additionally, the number of equations involved in the solution procedure can be sensibly reduced by avoiding to enforce unnecessary geometrical conditions.

Also, in our formulation of the method we do not exploit the duality with graphic solution procedures, in particular the correspondence between TNA and graphic statics used to determine the desired distribution of thrusts. Actually we will show that this graphical manipulations can be avoided by setting proper bounds to the thrust values used in the linear optimizations involved in the analysis.

Finally and most importantly, we consider the case in which also horizontal loads are applied to the network. Such an extension, only mentioned in previous formulations, can significantly complicate the evaluation of the thrust forces, thus prompting for a proper solution procedure to be developed.

2.1. Thrust networks

Equilibrium of vaulted structures can be studied by considering a network of thrusts, i.e. compressive forces acting within the structure in equilibrium with the applied loads. Such a network, from now on denominated *thrust network*, is described by means of N_n nodes and N_b branches connecting pairs of nodes.

We emphasize that the thrust network is not used to geometrically model the volume occupied by the vaulted structure, as it happens, e.g., in finite element modeling; rather it is representative of the thrust forces that equilibrate the external loadings. Accordingly, branches of the network represent the direction of the thrust forces, similarly to the branches of a funicular polygon.

The nth node of the network is characterized by its position (x_n, y_n, z_n), in a three-dimensional Cartesian reference frame in which z is the vertical direction. The generic branch b of the network is identified by two end nodes and the corresponding value of the thrust force, denoted as $\mathbf{t}^{(b)} = \left(t_x^{(b)}, t_y^{(b)}, t_z^{(b)}\right)$.

Nodes are loaded both by an external force $\mathbf{f}^{(n)} = \left(f_x^{(n)}, f_y^{(n)}, f_z^{(n)}\right)$, which value depends on the region of influence of the node, and by the thrust forces pertaining to branches connected to the node; if compressive, these thrust forces are oriented towards node n.

Branches can be labeled as *internal*, if they represent a thrust force that is interior to the network, *edge* if they represent forces that are on a free edge or *external* if they represent the support reactions, see, e.g., Fig. 1. Following the same logic, the set of nodes is split into N_i internal nodes, N_e edge nodes and N_r external (restrained) nodes, where only one external branch converges. Hence one has $N_n = N_i + N_e + N_r$.

While the horizontal position of internal and external nodes are assigned, the coordinates of the edge nodes are unknown. This is due to the fact that relevant edge branches, i.e. branches connected by edge nodes, will be funicular, in both the horizontal and vertical directions, of the internal thrusts and of the applied loads converging to edge nodes.

Equilibrium conditions are employed in order to evaluate branch thrusts, heights of internal and external nodes and the coordinates of boundary nodes. Such equations are written only for internal and boundary nodes, while external nodes are used uniquely as endpoints of external branches that, in turn, represent support reactions. Accordingly, external nodes and branches are used to model the constraints of the vaulted structure.

2.2. Horizontal equilibrium of nodes

The forces entering the horizontal equilibrium of the generic nth node of the network are the components $f_x^{(n)}$ and $f_y^{(n)}$ of the external loads and the horizontal components $t_x^{(b)}$ and $t_y^{(b)}$ of the thrust force relative to the branches b connected to the node. Accordingly, horizontal equilibrium enforces