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Exact elasticity-based finite element for circular plates

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ABSTRACT

In this paper, a general elasticity solution for the axisymmetric bending of a linearly elastic annular plate is used to derive an exact finite element for such a plate. We start by formulating an interior plate problem by employing Saint Venant's principle so that edge effects do not appear in the plate. Then the elasticity solution to the formulated interior problem is presented in terms of mid-surface variables so that it takes a form similar to conventional engineering plate theories. By using the mid-surface variables, the exact finite element is developed both by force- and energy-based approaches. The central, non-standard feature of the interior solution, and the finite element based on it, is that the interior stresses of the plate act as surface tractions on the plate edges and contribute to the total potential energy of the plate. Finally, analytical and numerical examples are presented using the elasticity solution and the derived finite element.

1. Introduction

It is relatively easy to derive general closed-form solutions for the axisymmetric bending of linearly elastic circular Kirchhoff, Mindlin and Levinson plates [1–3]. Furthermore, the analytical solutions may be used to develop exact, locking-free plate finite elements [4–6]. Closed-form elasticity solutions that relax the kinematic and constitutive assumptions of the aforementioned conventional plate theories, however, are not that well-known; the only solution found in standard textbooks is the one for a uniformly loaded simply-supported solid circular plate [7,8]. Thus, it comes as no surprise that exact finite elements for the axisymmetric bending of circular plates founded on closed-form elasticity solutions do not exist. In this paper, we develop such an element by using a suitable general elasticity solution.

The exact elasticity solution for the uniformly loaded simply-supported circular plate [7] is in fact an *interior* solution that excludes all edge effects by virtue of Saint Venant's principle. When a *full* solution is of interest, the most general state of stress within a linearly elastic, isotropic, homogeneous plate can be decomposed into three parts: (1) interior state, (2) shear state, and (3) Papkovich–Fadle state [9–11]. Detailed, general 3D elasticity solutions for plates which account for all three states have been given by several authors [12–17]. The shear and Papkovich–Fadle states are indeed predominantly related to edge effects, whereas the interior state represents the conventional "plate theory part" [9]. We use the general interior solution of Piltner [16], obtained by using displacement potentials, to derive an exact finite element for the axisymmetric bending of circular plates. General interior elasticity solutions derived using the Airy stress function have been recently used to develop finite elements for plane beams in a likewise manner [18,19].

The distinction between the three different stress states is highly important due to the fact that if our plate consists solely of the interior state, the interior stresses of the plate act as surface tractions on the lateral edges of the plate and, thus, they contribute significantly to the total potential energy of the plate. It is commonplace not to account for this property in energy-based treatments of plate theories founded exclusively on interior behavior. Therefore, the energy-based formulation of the plate finite element to be presented herein is fundamentally different from the conventional practices because we take the work due to the interior stresses on the plate edges properly into account. Although the present study is limited to linearly elastic, isotropic, homogeneous plates that undergo small deformations, the employed interior methodology is expected to find wider application in the study of engineering plate theories.

The remainder of this paper is organized in the following way. In Section 2, an interior problem is formulated for a circular plate and the implications of the interior definition are discussed. In Section 3, the general elasticity solution to the formulated problem is studied. Using mid-surface variables formed from the solution, an exact axisymmetric annular plate finite element is formulated in Section 4 both by force- and energy-based methods. A variety of

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analytical and numerical examples are presented in Section 5. Finally, conclusions are drawn in Section 6.

2. Interior problem formulation

2.1. Boundary conditions

A linearly elastic, isotropic, homogeneous annular plate under a rotationally symmetric transverse load p is shown in Fig. 1. The thickness of the plate is h and the outer and inner radii of the plate are a and b, respectively. The stress boundary conditions on the upper and lower faces of the plate read

$$\sigma_z(r,-h/2) = -p, \quad \sigma_z(r,h/2) = 0, \quad \tau_{rz}(r,\pm h/2) = 0. \tag{1}$$

The boundary conditions are introduced in a *strong* (pointwise) sense for the upper and lower faces. On the inner and outer lateral edges of the plate the tractions are specified through stress resultants as suggested by Fig. 1 and, thus, the boundary conditions on the lateral edges are imposed only in a *weak* sense [8]. The stress resultants per unit length are calculated from the equations

$$M_r = \int_{-h/2}^{h/2} \sigma_r z dz, \quad M_\theta = \int_{-h/2}^{h/2} \sigma_\theta z dz, \quad Q_r = \int_{-h/2}^{h/2} \tau_{rz} dz, \tag{2}$$

where $M_r(r)$ and $Q_r(r)$ are the radial bending moment and shear force, respectively, and $M_\theta(r)$ is the tangential bending moment. The positive directions of the radial stress resultants are given in Fig. 1.

The replacement of the strong stress boundary conditions along the plate edges by the statically equivalent weak boundary conditions (stress resultants) implies that all detailed, exponentially decaying edge effects of the plate are eliminated by virtue of Saint Venant's principle and only the interior solution of the plate is under consideration. The general homogeneous solution by Piltner [16] to the formulated interior plate problem for stress-free faces will be studied in Section 3. A uniformly distributed load will be added as the particular contribution to the solution.

2.2. Boundary layer and implications of the interior definition

Let us consider the solid circular plate with a boundary layer shown in Fig. 2. If pointwise boundary conditions were to be imposed on the outer edge of the boundary layer at $r = a'$, the detailed distributions of the resulting stresses would bring about edge effects which decay exponentially towards the interior of the plate. As a rule of thumb in isotropic cases, the boundary layer is as thick as the plate itself, that is, the thinner the plate is, the weaker the edge effects are. Studying a plate which consists only of an interior part means that the boundary layer has been removed. This amounts to fully-developed interior stresses being active all-over the plate at hand, including the lateral plate edge, where they act as surface tractions. In the case of an annular plate, an analogous discussion may be extended to the inner boundary region.

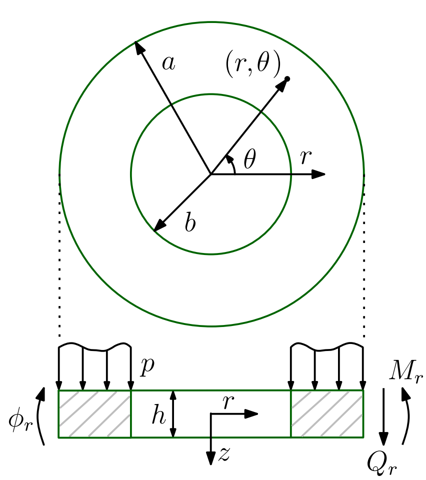

Fig. 1. Axisymmetric annular interior plate under a rotationally symmetric transverse load p on the upper face. The positive directions of the stress resultants and rotation ϕ_r on the outer edge are shown.

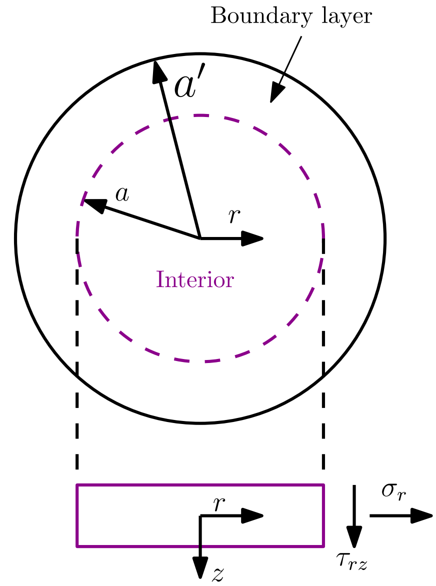

Fig. 2. Solid circular plate consisting of an interior part and a boundary layer. When only the interior plate is studied, stresses σ_r and τ_{rz} do work on the plate edge.

The key feature of the interior plate definition is that the interior stresses acting as surface tractions contribute to the total potential energy of the plate which reads

$$\Pi = U - W_p - W_s \tag{3}$$

where the strain energy for an annular plate is

$$U = \pi \int_b^a \int_{-h/2}^{h/2} r(\sigma_r\epsilon_r + \sigma_\theta\epsilon_\theta + \sigma_z\epsilon_z + \tau_{rz}\gamma_{rz}) dz dr \tag{4}$$

and the external work due to a uniformly distributed load $p = p_0$, which is of interest to us in the following sections, is given by

$$W_p = 2\pi \int_b^a r p_0 U_z(r,-h/2) dr. \tag{5}$$

The work by the surface tractions due to the interior stresses on the outer and inner lateral edges of the interior plate is given by

$$W_s = 2\pi a \int_{-h/2}^{h/2} [\sigma_r U_r + \tau_{rz} U_z](a,z) dz \\ -2\pi b \int_{-h/2}^{h/2} [\sigma_r U_r + \tau_{rz} U_z](b,z) dz. \tag{6}$$

where $U_r(r,z)$ and $U_z(r,z)$ are the displacements in the directions of r and z, respectively.

3. General interior elasticity solution

3.1. Homogeneous solution

Starting from Piltner's [16] solution for a rectangular plate, the general interior solution for a linearly elastic, isotropic,