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An efficient algorithm based on group theory and the Woodbury formula for the dynamic responses of periodic structures

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ABSTRACT

An efficient numerical method for computing the dynamic responses of periodic structures is proposed. Efficiently solving a system of linear equations is a key issue for computing the dynamic response of a large-scale structure. Based on the periodic properties of a structure and using condensation technology, the system of linear equations that corresponds to the periodic structure is reduced to a small-scale system of linear equations. Based on the Woodbury formula, the solution for the small-scale system of linear equations is obtained by solving a new system of linear equations whose coefficient matrix corresponds to a cycle periodic structure. Using group theory, the new system of linear equations is efficiently solved. Superior efficiency is achieved because the scale of the system of linear equations for the entire structure is significantly reduced and the coefficient matrix for the new system of linear equations can be converted into a block-diagonal matrix using group theory. Numerical examples are presented to illustrate the high efficiency of the proposed method.

1. Introduction

A periodic structure consists of a number of identical structural components (unit cells) that are repeated in one, two or three directions to form a complete system, such as a beam lattice [1,2], periodic beam structures [3,4], cellular solids [5], periodic composite structures [6–8], and photonic [9] and phononic [10,11] crystals. The theoretical studies and applications of the periodic structures have aroused substantial concern among research institutions and scholars all over the world.

The wave motion and propagation are very important aspects for the periodic structures. In this area, it is mainly concentrated on the dispersion relations and the scattering analysis of the periodic structures. The dispersion relation has attracted the attention of many scholars. Tassilly [12] derived the dispersion relation between wavenumber and frequency for a class of non-uniform periodic elements. Gavric [13] used the concepts of cross-section modes and finite element method to calculate the dispersion relation for an free rail. Using the transfer matrix method and Floquet's theorem, Shen and Cao [14] derived a dispersion relation for acoustic wave propagation in a periodic layered structure. For the infinite two-dimensional periodic lattices, the dispersion curves were obtained by solving the eigenvalue problem for wave propagation [15]. By studying the wave behavior in periodically simply supported beam and periodic frame structures, Sonekar and Mitra [16] analyzed the dispersion relation for the periodic structures. Through analysis of the elastic wave propagation in one-dimensional periodic elastic rod structure and two-dimensional periodic elastic beam structure, Tian et al. [17] derived the dispersion relation between Bloch wave vectors and eigenfrequencies. Trainiti et al. [18] investigated the effects of periodic geometric undulations on the dispersion properties of one-dimensional and two-dimensional elastic structures. Meanwhile, the scattering analysis has also received much attention. Vonflotow [19] showed the scattering behavior of the junction of the Timoshenko bending model and the periodic torsion model. Cai and Lin [20] analyzed wave scattering in generic structural networks. By studying the propagation properties of flexural wave in the periodic beam on elastic foundations, Yu et al. [21] designed and calculated Bragg scattering gap. For analyzing the scattering in infinite periodic structures, Petersson and Jin [22] proposed a new two-dimensional time domain finite element method formulation.

Dynamic vibration is another important aspect for the periodic structures. In this area, it is mainly concentrated on the energy band and dynamic responses of the periodic structures. Because the band gap of periodic structure could be exploited in devices

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with widespread application in engineering, it has been received much attention. Schmidt and Kauf [23] computed the band structure of two-dimensional photonic crystals. Olhoff et al. [24] computed the maximizing frequency gaps of beam structures. Wang et al. [11] investigated the wave band gaps in two-dimensional piezoelectric and piezomagnetic phononic crystals. By employing the Bloch-Floquet theorem, Xiang and Shi [25] determined the flexural vibration band gaps in periodic beams. The vibration band-gap characteristics of periodic rigid frame structures composed of Timoshenko beams [26] and periodic Mindlin plate structure with two simply supported opposite edges [27] were studied. Meanwhile, solving the dynamic responses also have been attracted many attentions. Engels [28] investigated the dynamic responses of infinite and semi-infinite periodic structures to harmonic loads. Wu et al. [29] analyzed the dynamic behavior of periodic plate structures. Cai et al. [30] applied a U-transformation method to obtain dynamic solutions for periodic structures. Zhou et al. proposed efficient numerical approaches for studying the vibration of complex one-dimensional [31] and two-dimensional [32] periodic structures. Hawreliak et al. [33] investigated the dynamic responses of additively manufactured engineered lattice materials. Luongo and Romeo [34] presented a modified version of the traditional wave vector computational scheme for the dynamic analysis of long undamped periodic structures. Using a combination of wave and finite element approaches, Duhamel et al. [35] presented a method for calculating the force responses of periodic structures. Mencik [36] investigated a wave finite element method for computing the low- and mid-frequency forced responses of straight elastic structures. Based on a precise integration method, Gao et al. proposed efficient algorithms for computing the dynamic responses of one-dimensional [37] and two-dimensional [38] periodic structures. Gao et al. [39] derived an exact analytical solution for the dynamic response of a periodic structure for which the unit cell consists of one mass and one spring. Wu et al. [40] developed a subdomain precise integration method for the dynamic responses of periodic structures.

For large scale periodic structures, improving the computational efficiency of the numerical methods for solving the dynamic responses is an essential issue. It is well known that the group theory is an effective tool for improving the computational efficiency for symmetrical structures and so has been widely used in science and engineering [41–48]. Using the group-theoretic methods and substructuring technique, Zhong and Qiu [41] analyzed the symmetric or partially symmetric structures subjected to arbitrary loads. Based on the symmetry groups and representation theory, Zingoni [42] proposed an efficient computational scheme for the linear vibration analysis of high tension cable nets. Based on the full symmetry properties of a structure, group representation theory and Fourier method, the full stiffness matrix of a structure was converted into the form of block diagonalization [43], and the similar techniques was used to block-diagonalise the equilibrium matrix of a symmetric structure [44]. The unsymmetrical (or weakly symmetrical) spring-mass dynamic systems was transformed into equivalent dynamic systems featuring maximum possible symmetry, and then the group theoretic procedure employed to calculate all the eigenvalues of the systems [45]. Using the group theory, the eigenvalue problems were decomposed into independent subproblems due to the block diagonalized [46,47]. Zingoni [48] showed that the group-theoretic approach enables considerable simplifications and reductions in computational effort. However, although the infinite periodic structure is translational symmetry, but the periodic structure composed of finite unit cells has no symmetric property, so the group theory cannot be applied directly to the finite periodic structures.

This paper proposed an efficient algorithm for solving the dynamic responses of one-dimensional and two-dimensional periodic structures. The basic idea of the numerical algorithms for computing the dynamic responses is to transform the dynamic equations into a system of linear equations. Therefore, efficiently solving the system of linear equations is a key issue for computing the dynamic responses. Firstly, based on the periodic properties of a structure and using condensation technology, the system of linear equations for the entire periodic structure is reduced to a small-scale system of linear equations in this paper. Then, based on the Woodbury formula, the solution of the small-scale system of linear equations is obtained by solving a new system of linear equations whose coefficient matrix corresponds to a cycle periodic structure. Finally, using group theory, the new system of linear equations is efficiently solved. Using the three technologies enables an efficient algorithm to be proposed for solving the system of linear equations. This algorithm leads to an efficient method for computing the dynamic responses of periodic structures.

2. The basic equations

Assume that a structure is considered and that its stiffness, mass and damping matrices are $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{C}$, respectively. The dynamic equation can be written as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \tag{1}$$

where $\mathbf{f}$ denotes the external force vector, and $\mathbf{u}$, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ denote the displacement vector, velocity vector and acceleration vector, respectively.

Many numerical algorithms, such as the Newmark method [49], the generalized α method [50], the Bathe method [51,52], and the central difference method [53], can be applied for solving Eq. (1). For all methods, the basic idea is to transform Eq. (1) into the following system of linear equations

$$\bar{\mathbf{K}}\mathbf{u}_{t_0+\Delta t} = \bar{\mathbf{f}} \tag{2}$$

where $\bar{\mathbf{K}}$ and $\bar{\mathbf{f}}$ denote the equivalent stiffness matrix and the equivalent external force vector. Solving Eq. (2) gives the dynamic responses of Eq. (1).

The main computational effort for solving the structural dynamic responses comprises repeatedly performing the system of linear Eq. (2). For periodic structures, the degrees of freedom (DOF) of the entire structure is significant when it contains many identical unit cells. Thus, it is very time-consuming for solving the system of linear equations. Based on the characteristics of a periodic structure, an efficient method is proposed in this paper to solve the system of linear equations (2) using the Woodbury formula and group theory.

Because all previously mentioned numerical methods are required to solve Eq. (2), the Newmark method is used as an example in this paper to explain the idea of the proposed method. For the Newmark method, the equivalent stiffness matrix $\bar{\mathbf{K}}$ and equivalent external force vector $\bar{\mathbf{f}}$ are [49]

$$\begin{aligned}\bar{\mathbf{K}} &= \mathbf{K} + c_0\mathbf{M} + c_1\mathbf{C}\\ \bar{\mathbf{f}} &= \mathbf{f}_{t_0+\Delta t} + \mathbf{M}(c_0\mathbf{u}_{t_0} + c_2\dot{\mathbf{u}}_{t_0} + c_3\ddot{\mathbf{u}}_{t_0}) + \mathbf{C}(c_1\mathbf{u}_{t_0} + c_4\dot{\mathbf{u}}_{t_0} + c_5\ddot{\mathbf{u}}_{t_0})\end{aligned} \tag{3}$$

where $\mathbf{u}_{t_0}$, $\dot{\mathbf{u}}_{t_0}$ and $\ddot{\mathbf{u}}_{t_0}$ are the displacement vector, velocity vector and acceleration vector at t_0, $\mathbf{f}_{t_0+\Delta t}$ denotes the external force vector at $t_0 + \Delta t$ and c_0–c_5 in Eq. (3) are the parameters of the Newmark method [49]. Then, the displacement vector $\mathbf{u}_{t_0+\Delta t}$ at $t_0 + \Delta t$ can be obtained by solving Eq. (2), and the velocity and acceleration vectors at $t_0 + \Delta t$ can be obtained [49].