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# Analytical local and global sensitivity of power spectrum density functions for structures subject to stochastic excitation

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ABSTRACT

A unified consideration of the computation of the local and global sensitivity of structural Power Spectral Density (PSD) functions for random responses is conducted utilizing the Pseudo-Excitation Method (PEM). Analytical formulae enabling the calculation of sensitivity of the first and the second order with respect to chosen parameters are derived on the basis of an efficient direct algebraic method. The analytical formulae are compact, devoid of modal truncation problem and efficient to program. Also, global sensitivity analysis is implemented analytically by incorporating Gaussian process model to provide an approximate relationship between PSD functions and the sensitive parameters. It is investigated how scattering of parameters could induce variation in response spectra from a more comprehensive viewpoint. The validity and efficiency of the presented methods are illustrated using a numerical example.



## 1. Introduction

Power Spectral Density (PSD) functions are viewed as important physical quantities for structures subjected to stochastic excitations, and they provide an abundance of information beneficial for many applications [1–4]. Structural models usually involve a significant number of parameters to be investigated. As a result, it is of interest to determine how PSD change as structural model parameters are varied, thereby identifying parameters exerting the most influence on PSD, which are commonly known as 'sensitivity analysis' [5,6]. Sensitivity analysis with respect to structural parameters plays a fundamental role in its successful design, control and identification [7]. Sensitivity analysis can be divided into 'local sensitivity' and 'global sensitivity'. The former is concerned with the partial derivative of the PSD with respect to the model parameters evaluated at a specific point, while the latter deals with the variation of the PSD over a large region of parameter values [5].

Several contributions have been devoted to studying local sensitivity of PSD. Among others, Marano et al. presented a stochastic approach to define response spectra of a single-degree-of-freedom system subjected to a nonstationary seismic excitation, following which a new sensitivity analysis methodology was developed [8]. In double frequency domain, the formulation of analytical sensitivity statistics of various dynamic response quantities with respect to structural parameters was developed to tackle non-stationary earthquake motion in [9]. A numerical method for calculating the sensitivity and Hessian matrix of the response evolutionary PSD functions of structures subjected to evolutionary random seismic excitation was proposed in [10,11]. The methods were formulated based on the Pseudo-Excitation Method (PEM) [12] accompanied by numerical integration methods such as the Gauss precise time step approach or Newmark-β approach. More recently, three analytical methods were proposed to calculate the sensitivity and Hessian matrix of PSD functions for stationary random seismic responses [1]. The formulas were derived on the basis of Complete Quadratic Combination method, Square Root of the Sum of Squares method and PEM. This study highlights the advantages of PEM in computing the sensitivity of PSD under stochastic excitation. The method proposed by Liu to calculate the sensitivity of PSD functions in [1] is new, while its derivations are rigorous.

Global sensitivity analysis (GSA) (commonly called variance-based method) investigates the effects of simultaneous parameter variations in the entire allowable ranges and interaction effects among parameters on the model outputs. Measuring the importance for model parameters subjected to uncertainty, GSA maintains a number of advantages over local sensitivity approaches, including model independence, capability to account for interaction effect, ability to evaluate the influence of the whole parameter space, and being able to tackle groups of parameters [13]. The GSA is powerful and it has been widely explored in a variety of fields [14–17]. Although GSA is a powerful tool to assess the impact of uncertain parameters on model outputs, it remains computation-

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ally intensive, especially when applied to the large-scale and complex systems [18].

It is worth mentioning that, for both local sensitivity and global sensitivity analysis of PSD, there are some issues required to be addressed. For local sensitivity analysis, it involves calculating the sensitivity of frequency and mode shapes, and the eigenvector derivatives are expressed as a linear combination of the full set of eigenvectors in [1]. Such treatment suffers from the problems of modal truncation errors and possible programming inefficiency. Regarding GSA of PSD, it is still a problem rarely reported ever before. More importantly, the high computational cost involved in GSA restricts the usage of traditional methods such as Monte Carlo simulation (MCS) due to the expensive-to-run model.

To address the issues mentioned in the above, a unified consideration of the computation of the local and global sensitivity analysis of PSD for stationary stochastic responses is implemented in this study. Motivated along the direction of [1], the computational efficiency and computational accuracy are enhanced by borrowing the novel idea of PEM. Furthermore, by incorporating direct algebraic method, analytical formulae are derived to calculate the local sensitivity of the first and the second order with respect to chosen parameters. The formulae are compact, devoid of modal truncation problem and efficient to program. To circumvent the issue of high computational expense, a Gaussian process model (GPM)-based analytical method [19] is adopted to conduct GSA of PSD in an efficient manner. The validity of the presented methods is illustrated using a numerical example.

## 2. Theoretical basis

### 2.1. Governing equilibrium equation

Consider a linear structure with $n_d$ dofs subjected to the excitation of the ground horizontal acceleration time history and the governing equilibrium equation of motion with generalized coordinates $\mathbf{x}(t)$ are given by

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=-\mathbf{M}\mathbf{E}_u\ddot{x}_g(t) \tag{1}$$

where $\mathbf{M}\in\mathbb{R}^{n_d\times n_d}$, $\mathbf{C}\in\mathbb{R}^{n_d\times n_d}$ and $\mathbf{K}\in\mathbb{R}^{n_d\times n_d}$ are the mass, damping and stiffness matrices; $\mathbf{E}_u$ is a force distributing vector composed of zeros and ones, and $\ddot{x}_g(t)$ is ground motion. Using the normal coordinate transformation, the generalized coordinates $\mathbf{x}(t)$ can be expressed as

$$\mathbf{x}(t)=\sum_{j=1}^{n}u_j(t)\mathbf{\Phi}_j \tag{2}$$

where $\mathbf{\Phi}_j$ denotes the $j$-th eigenvector of the structure, which is normalized to be unit-mass mode shapes, i.e. $\mathbf{\Phi}_j^T\mathbf{M}\mathbf{\Phi}_j=1$; $n$ denotes the number of modes, which are usually less than the number of degrees of freedom $n_d$ in real applications. Rayleigh damping is assumed here with $\mathbf{C}=\alpha_0\mathbf{M}+\alpha_1\mathbf{K}$ where $\alpha_0=2\omega_1\omega_2(\zeta_1\omega_2-\omega_1\zeta_2)/(\omega_2^2-\omega_1^2)$ and $\alpha_1=2(\zeta_2\omega_2-\zeta_1\omega_1)/(\omega_2^2-\omega_1^2)$. Here $\omega_1$ and $\omega_2$ are the first and the second natural frequency of the structure concerned, respectively, while $\zeta_1$ and $\zeta_2$ are the first and second damping ratios, respectively. As a result, (1) can be decoupled to $n$ separate equations for each of the modes. For the $j$-th mode, one has

$$\ddot{u}_j(t)+2\zeta_j\omega_j\dot{u}_j(t)+\omega_j^2u_j(t)=-\gamma_j\ddot{x}_g(t) \tag{3}$$

where $\zeta_j$ is the $j$-th damping ratio, and $\gamma_j$ is the $j$-th mode participation factor given by

$$\gamma_j=\mathbf{\Phi}_j^T\mathbf{M}\mathbf{E}_u \tag{4}$$

### 2.2. Pseudo excitation method

In Liu's work [1], PEM was directly employed to derive the formulae of the first and second derivatives of PSD functions. Our work will also borrow the concept from [1]. Therefore, the general concept of PEM will be outlined in this section. Interested readers are referred to [12]. Using PEM, stationary random response analysis as the primary concern of our study can be converted into harmonic response analysis. The pseudo excitation of ground acceleration is constructed as follows

$$\ddot{\tilde{x}}_g(t)=\sqrt{S_{\ddot{x}_g(t)}}e^{\mathbf{i}\omega t} \tag{5}$$

For the $j$-th mode, one has

$$\ddot{\tilde{u}}_j(t)+2\zeta_j\omega_j\dot{\tilde{u}}_j(t)+\tilde{u}_j(t)=-\gamma_j\sqrt{S_{\ddot{x}_g}(\omega)}e^{\mathbf{i}\omega t} \tag{6}$$

The stationary solution of $\tilde{u}_j(t)$ is equal to

$$\tilde{u}_j(t)=-\gamma_jH_j\sqrt{S_{\ddot{x}_g}(\omega)}e^{\mathbf{i}\omega t} \tag{7}$$

where $H_j(\omega)$ is the frequency response function of the $j$-th mode, i.e.

$$H_j(\omega)=\left(\omega_j^2+2\zeta_j\omega_j\omega\mathbf{i}-\omega^2\right)^{-1} \tag{8}$$

According to the PEM, the PSD matrix of the real displacement vector is equal to

$$\begin{aligned}\mathbf{S}_{\mathbf{x}}(\omega)&=\tilde{\mathbf{x}}(t)\tilde{\mathbf{x}}^*(t)\\&=\left(\sum_{j=1}^{n}\gamma_jH_j(\omega)\mathbf{\Phi}_j\sqrt{S_{\ddot{x}_g}(\omega)}e^{\mathbf{i}\omega t}\right)\left(\sum_{j=1}^{n}\gamma_jH_j(\omega)\mathbf{\Phi}_j\sqrt{S_{\ddot{x}_g}(\omega)}e^{\mathbf{i}\omega t}\right)^*\end{aligned} \tag{9}$$

Denoting that

$$\mathbf{\Theta}_j(\omega)=\gamma_jH_j(\omega)\mathbf{\Phi}_j \tag{10}$$

(9) can be rearranged as

$$\mathbf{S}_{\mathbf{x}}(\omega)=S_{\ddot{x}_g}(\omega)\left(\sum_{j=1}^{n}\mathbf{\Theta}_j(\omega)\right)^*\left(\sum_{j=1}^{n}\mathbf{\Theta}_j(\omega)\right)^T \tag{11}$$

## 3. Local sensitivity analysis

### 3.1. Eigen-sensitivity

To calculate the sensitivity and the Hessian of PSD functions, the first-order derivatives and the second-order derivatives of eigenpair have to be known. For a linear and undamped structural system with $m$ elements and $n_d$ degrees of freedom, the expressions of the first-order derivatives of eigenvalue and eigenvector sensitivity with respect to design variable $p$ are given by [20–24]

$$\boldsymbol{\mu}_j^{(p)}=\begin{Bmatrix}\mathbf{\Phi}_j^{(p)}\\ \lambda_j^{(p)}\end{Bmatrix}=\mathbf{K}_{\mathbf{U/D}}^{-1}\mathbf{\Lambda}_j \tag{12}$$

where $\mathbf{K}_{\mathbf{U/D}}=\begin{bmatrix}\mathbf{K}-\lambda_j\mathbf{M} & -\mathbf{M}\mathbf{\Phi}_j\\ -\mathbf{\Phi}_j^T\mathbf{M} & 0\end{bmatrix}$ and $\mathbf{\Lambda}_j=\begin{Bmatrix}-(\mathbf{K}^{(p)}-\lambda_j\mathbf{M}^{(p)})\mathbf{\Phi}_j\\ \frac{1}{2}\mathbf{\Phi}_j^T\mathbf{M}^{(p)}\mathbf{\Phi}_j\end{Bmatrix}$. Here the superscript denotes differentiation. $\lambda_j=\omega_j^2$ denotes the $j$-th eigenvalue.

As derived in Appendix A, one can obtain $\boldsymbol{\mu}_j^{(pq)}$ by differentiating $\boldsymbol{\mu}_j^{(p)}$ with respect to design variable $q$,

$$\boldsymbol{\mu}_j^{(pq)}=\begin{Bmatrix}\mathbf{\Phi}_j^{(pq)}\\ \lambda_j^{(pq)}\end{Bmatrix}=-\mathbf{K}_{U/D}^{-1}\mathbf{K}_{U/D}^{(q)}\mathbf{K}_{U/D}^{-1}\mathbf{\Lambda}_j+\mathbf{K}_{\mathbf{U/D}}^{-1}\mathbf{\Lambda}_j^{(q)} \tag{13}$$