Computers and Structures 182 (2017) 337–353

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

A novel high-performance mixed membrane finite element for the analysis of inelastic structures

Nicola A. Nodargi *, Paolo Bisegna

Department of Civil Engineering and Computer Science, University of Rome "Tor Vergata", 00133 Rome, Italy

ARTICLE INFO

Article history:
Received 4 May 2016
Accepted 4 October 2016

Keywords:
Quadrilateral elements
Mixed formulation
Material nonlinearity
Discontinuous interpolation
Element state determination

ABSTRACT

A novel high-performance eight-node quadrilateral mixed finite element is presented. Its formulation is based on a Hu-Washizu type functional, suitable to the treatment of material nonlinearities. In order to capture, even for coarse meshes, highly nonlinear spatial distribution of strain field, strain interpolation is assumed to be discontinuous, piecewise-constant over suitable element subdomains. A robust element state determination procedure is proposed to solve compatibility and constitutive equations at element level. The mixed element stability is numerically assessed. Supported by the comparison with compatible quadrilaterals, numerical simulations concerning elastoplastic and shape-memory alloy structures prove accuracy and effectiveness of the proposed formulation.

1. Introduction

Analysis of inelastic structures, as required for design and safety assessment procedures in engineering problems, represents a central task in computational mechanics. Thus, the arising challenge is to conceive a high-performance finite element formulation that combines accuracy (even for coarse meshes) together with robustness and limited computational cost.

Unfortunately, the extension to materially non-linear context of standard displacement-based (or compatible) formulations is unable to meet those needs (for instance, see [1]). In fact, in such a formulation, the displacement field is approximated by a polynomial interpolation and the corresponding strain field is determined by the compatibility condition in strong form, i.e. by differentiation. Accordingly, computationally expensive fine meshes are required for polynomial functions to capture the possibly highly nonlinear strain spatial distribution, thus reducing the efficiency of the approach. Those complications with compatible formulations prompted the interest of researchers towards mixed finite elements (for a detailed account on the topic, see [2]), originally introduced to tackle problems characterized by some physical constraint, such as the incompressible or nearly incompressible behavior of rubber-like media, or the shear constraint in plate problems (e.g., see [3]). In particular, the main advantages of mixed formulations are [4]: (i) *relaxed continuity requirements*, achieved by approximating as primary unknowns not only the displacement, but also the strain and/or stress fields, (ii) *better stress solution*, computed without differentiation of displacement field, and (iii) *better displacement solution*, also for coarse and irregular meshes.

The first mixed quadrilateral element for plane problems was proposed by Pian and Sumihara [5] and consisted in a Hellinger-Reissner (HR) formulation with bilinear interpolation of displacement field and linear interpolation of stress field. To date, several potential improvements have been sought for. Among those, the introduction of corner rotational degrees of freedom (DOFs), initially explored by Allman [6], has been recognized as an attractive strategy to pursue intermediate accuracy between linear and quadratic elements with translations only, even though introducing zero-energy spurious modes. Notable HR formulations of four-node quadrilaterals equipped with drilling rotations are the elements HQ4-9β [7] and HS-A7 [4]. Both of them are characterized by stress interpolations which satisfy internal equilibrium in strong form, and consequently require the approximation of the displacement field only over the element boundary. Remarkable features are the adoption of Allman-type interpolation for side displacement plus a stabilization of the arising zero-energy spurious mode [7], and the derivation of self-equilibrated stress approximation resorting to the Airy's function approach [4]. A not complete list of other proposed four-node quadrilaterals with drilling rotations comprises strain-enhanced formulations [8], applications of Trefftz method [9,10] and strain-based models [11]. Alternatively to the introduction of rotational DOFs, improved accuracy can be achieved by using eight-node quadrilaterals, such as in the HR formulations HBQ8 [12] and HSF-Q8-15β [13]. In particular, in the latter the stress interpolation is self-equilibrated and the

* Corresponding author.
E-mail address: nodargi@ing.uniroma2.it (N.A. Nodargi).

http://dx.doi.org/10.1016/j.compstruc.2016.10.002
0045-7949/

boundary displacement is approximated by the restriction of isoparametric serendipity element shape functions. A formulation which aims at combining the advantages of eight-node quadrilaterals with corner rotational DOFs is the element HQ8-13β, presented in [14]. Indeed, it can be regarded as a generalization of element HQ4-9β, obtained by assigning a tangential DOF to the mid-side nodes.

Beside the aforementioned mixed formulations, not specifically pertaining to inelastic structures, attempts of designing high-performance finite elements in elastoplasticity trace back to Maier's work [15,16] and his notion of *constitutive law of the finite element*, resulting from the imposition of the constitutive equations in a discretized form at finite element level. Following such approach, Comi and Perego [17] presented a Hu-Washizu (HW) formulation of the elastoplatic boundary value problem, entailing weak form expression of the constitutive equations. More recently, Bilotta and coauthors [18–20] exploited a HR functional in which the plastic multiplier is assumed as an independent variable (and interpolated over the finite element mesh) to formulate four-node quadrilateral elements. Therein, the imposition of the constitutive law at finite element level amounts to solve a problem analogous to the state update of an elastoplastic material, and is consequently addressed by means of a return mapping procedure by sequential quadratic programming (for an example of convex programming in elastoplasticity, see [21]). A similar formulation, although referred to an eight-node quadrilateral element, is also adopted in [22], where the traditional von Mises yield surface is piecewise linearized and the element constitutive equations are recast in a linear programming problem.

In the present work, a new high-performance eight-node membrane quadrilateral mixed element is proposed for the analysis of inelastic structures. For the formulation to have a well-posed variational foundation, the material is supposed to be endowed with a strictly convex stress potential, such as a Helmholtz free energy for hyperelastic materials or a suitable incremental energy for generalized standard materials with hardening behavior [23–26]. The element derivation is based on a HW type functional, suitably modified to exploit stress interpolation satisfying internal equilibrium in strong form. The boundary displacement is interpolated by quadratic Lagrangian shape functions, that is, the restriction of isoparametric serendipity element shape functions to the element boundary [14]. The self-equilibrated stress interpolation is generated through the Airy's function approach and consists in a second-order complete basis, enriched by two cubic modes. Therefore, the resulting element is isostatic, i.e. possesses the minimum number of stress modes to suppress all deformational ones.

Distinctive feature of the present formulation is the strain interpolation, assumed to be discontinuous, piecewise-constant over suitable element subdomains. This choice is the counterpart of the discontinuous strain interpolation implicitly used in [27] and actually recognized in [1] for beam elements. When considering inelastic materials, internal variables (for instance, plastic strain in an elastoplastic medium) are sampled at quadrature points, and may exhibit highly nonlinear spatial distribution. The latter is inherited by the strain field, since elastic strain usually exhibits a more regular distribution, related to equilibrated stress distribution. Thus, for better reproducing the possibly highly nonlinear spatial distribution of the strain field, it seems reasonable to assume discontinuous, piecewise-constant interpolation for the strain as well. On the other hand, the present strain interpolation seems to be a generalization of the approach proposed for elastoplastic media in [18–20], allowing a unified treatment of inelastic materials within the class of generalized standard ones.

The static condensation of strain and stress parameters at element level is performed, which is legitimate because the relevant interpolations are element supported. It requires to solve compatibility and constitutive equations. However, due to nonsmooth material nonlinearity, this goal is difficult to accomplish through a direct application of Newton's method, even if in conjunction with line-search techniques. Here an efficient and robust iterative procedure, similar to the ones discussed in [1,27–30], is proposed to perform such element state determination. The basic idea of the algorithm is to regard element equations as a function mapping nodal forces onto nodal displacements, up to a rigid body motion. Within this setting, the element state determination amounts to the inversion of this map for the element nodal displacements at the current structural iteration, and Newton's method can be successfully applied to this end. More specifically, given an estimation of the nodal internal-force vector, corresponding stresses are determined using equilibrium equations and then element subdomain constitutive relationships are solved for strains. In case the latter are compatible with given nodal displacements, the procedure ends, otherwise an updated estimate of nodal internal-force vector is computed and the algorithm proceeds until convergence. The crucial advantage of the proposed procedure is the possibility to solve independently from each other as many material state update problems as the number of element subdomains, thus reducing the computational cost which stems from coupled nonlinear constitutive equations, and mitigating convergence difficulties of Newton's method.

The element mixed formulation is concluded with stability analysis, successfully addressed through the simple and reliable numerical test proposed in [31]. Finally, numerical simulations are reported for assessing accuracy, robustness and effectiveness of the proposed quadrilateral, and comparing its performances with compatible quadrilaterals. In particular, the capability of the present formulation to unify the treatment of inelastic materials equipped with a stress potential, is shown in the analyses of structures composed of elastoplastic or shape-memory alloy (SMA) materials.

The paper is organized as follows. In Section 2 the variational formulation and its mixed finite element approximation are discussed. The interpolation spaces for unknown variables are selected in Section 3. In Section 4 the element state determination procedure is described, whereas in Section 5 the numerical test used to explore the element stability is presented. Numerical simulations are presented in Section 6, and conclusions are outlined in Section 7. Some complementary results are reported in appendices: a closed form expression for the proposed element compatibility matrix is presented in Appendix A, filtering out rigid body motions from element nodal DOFs is presented in Appendix B, whereas details on the constitutive law inversion for elastoplastic material with von Mises yield function or shape-memory alloy material are given in Appendix C.

2. Variational formulation

2.1. Generalized standard materials

In the framework of small deformation theory, the constitutive behavior of a generalized standard material [23] is described in terms of a strictly convex Helmholtz free energy φ, depending on the strain $\boldsymbol{\varepsilon}$ and on a generalized vector of internal variables $\mathcal{I}$, and in terms of a dissipation potential D, which is supposed to be a function of the flux of the internal variables $\dot{\mathcal{I}}$. By standard thermodynamic arguments, the constitutive equations for the stress $\boldsymbol{\sigma}$ and the generalized vector of internal forces $\mathcal{F}$ conjugated to internal variables $\mathcal{I}$ turn out to be:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon},\mathcal{I}) = \partial_{\boldsymbol{\varepsilon}}\varphi(\boldsymbol{\varepsilon},\mathcal{I}), \quad \mathcal{F}(\boldsymbol{\varepsilon},\mathcal{I}) = -\partial_{\mathcal{I}}\varphi(\boldsymbol{\varepsilon},\mathcal{I}), \tag{1}$$

where ∂ denotes the sub-differential operator of convex functions. The evolution in time of the internal variables $\mathcal{I}$ is governed by