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Enriched finite elements for initial-value problem of transverse electromagnetic waves in time domain

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ABSTRACT

This paper proposes a partition of unity enrichment scheme for the solution of the electromagnetic wave equation in the time domain. A discretization scheme in time is implemented to render implicit solutions of systems of equations possible. The scheme allows for calculation of the field values at different time steps in an iterative fashion. The spatial grid is partitioned into a finite number of elements with intrinsic shape functions to form the bases of solution. Furthermore, each finite element degree of freedom is expanded into a sum of a slowly varying term and a combination of highly oscillatory functions. The combination consists of plane waves propagating in multiple directions, with a fixed frequency. This significantly reduces the number of degrees of freedom required to discretize the unknown field, without compromising on the accuracy or allowed tolerance in the errors, as compared to that of other enriched FEM approaches. Also, this considerably reduces the computational costs in terms of memory and processing time. Parametric studies, presented herein, confirm the robustness and efficiency of the proposed method and the advantages compared to another enrichment method.

1. Introduction

We live in an age in which we have harnessed electromagnetic waves to engineer a wide variety of products and systems on which modern societies have come to rely. Medical imaging devices, mobile communications and electrical power generation are just a few examples of technologies that are entirely reliant on electromagnetic phenomena. While the underlying differential equations that govern these phenomena were developed in the 19th century, their application to realistic engineering problems requires numerical approximations, and engineers continue to develop more advanced computational methodologies capable of delivering these approximations with higher fidelity and with efficient use of computational resources. In this paper we specifically address short wave problems, which is of increasing importance with the prospect of moving to millimetre wave technologies for 5G wireless systems.

We confine ourselves in this discussion to the deterministic methods, i.e. those giving a unique solution given a well-posed problem subject to prescribed boundary and initial conditions. Common numerical methods include the Finite Difference Method (FDM) [32,33,36] the Finite Element Method (FEM) [6,15,28] and the Boundary Element Method [3–5]. Among these methods, the FEM is well established for dealing with complicated geometries or inhomogeneous media, and the other methods also offer certain advantages, but they all remain constrained in term of the problem size. This is mainly due to the fact that the computational domain may be very large (electromagnetic waves often propagate in free space), so that the size of the analysis domain and of scattering objects can greatly exceed the wavelength, typically by multiple orders of magnitude [8,35]. Since a certain number of degrees of freedom are required to capture the solution over each wavelength, such problems can result in a very large system of equations. This can render them completely intractable using conventional FEM, FDM and BEM methodologies. Different authors vary in their recommendations, but a typical rule of thumb suggests the use of ten degrees of freedom in each wavelength for linear elements. To illustrate the problem, engineers may seek to analyse the scattering of a radar wave by an aircraft. Even if the analysis domain is confined to a 100 m cube surrounding the aircraft, a finite element model would require at least 10^{12} degrees of freedom to model the scattering of a radar wave of 100 mm wavelength to engineering accuracy.

To overcome this limitation without compromising the accuracy, the Partition of Unity enrichment method was proposed in

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[21] for harmonic wave problems governed by the Helmholtz equation. The method consists of enriching the approximation space with oscillatory functions that have better approximation properties compared to the standard low order polynomials usually used in the FEM. The enrichment idea spawned a large body of literature including the work on the Partition of Unity Finite Element Method (PUFEM) [7,18,19,22] and also similar enrichment techniques such as the Generalised Finite Element Method [30,31] the Ultra-Weak Variational Formulation [14,20] and the Discontinuous Enrichment Method [9,16,34]. The enrichment approach is also used in other methods such as the Boundary Element Method [25–27]. A recent survey on various enrichment approaches could be found in [12].

The enrichment functions used in the case of harmonic wave problems, as presented above, are in the form of plane waves or radial waves and are solutions of the partial differential equations (PDEs) governing the problems. This latter aspect, while useful, is not necessary for incorporating *a – priori* knowledge of the solution behaviour in the approximating field. This inspired the use of intuitive field enrichment functions capable of capturing the solution behaviour while not necessarily being solutions of the problem PDE. Such enrichment was proposed for solving heat transfer problems in the time domain [23,24,29]. The temperature field was enriched with Gaussian functions capable of modelling the high temperature gradients and led to the use of coarse mesh grids, instead of the very fine meshes employed in standard FEM, and hence to considerable savings in the computational effort. In spite of the problem being time dependent, the enrichment functions are independent of time, which permits the re-use of a single system matrix for all time steps, resulting in even further computational saving. The success of this approach has motivated the current work in developing the field-enrichment technique to solve time-dependent wave problems. It is worth noting that an enriched model for wave propagation in one dimensional problems was presented in [17]. Recently this was extended into two-dimensional transient wave problems [10] where the solution field within each finite element is discretized with the usual Lagrangian functions and enriched with harmonic functions, each with a prescribed frequency. In the current work, the PUFEM is used for the first time to solve the wave equation in the time domain. In previous work on the PUFEM only time harmonic problems were considered when solving the equation in the frequency domain. Instead here we show that the method could also be used for solving non-time-harmonic problems in the time domain. The wave field solution is presented as a sum of a slowly varying term and a highly oscillatory part, which is expanded into a sum of plane waves propagating in multiple directions. However, unlike the other enrichment technique [10] here the plane waves have a fixed frequency. The performance of this approach is assessed for different test wave models where exact solutions are available. The results are compared to those obtained by a polynomial based FEM and also to another enrichment approach where the proposed scheme provides better accuracy at a reduced computational cost.

The paper is organised as follows. In Section 2 we introduce the considered problem. Then in Section 3 we present the proposed PUFEM model, while in Section 4 the model is validated on two numerical test cases with analytical solutions. We finish with some concluding remarks and recommendations for future work in Section 5.

2. Transverse electric mode of propagation

To describe the techniques used for enriched finite elements we consider a linear wave equation in two-dimensional domains. Hence, let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz continuous boundary Γ and let $[0, T]$ be the time interval for the wave propagation. The boundary-value problem considered in the current study is defined as

$$\frac{\partial^2 E}{\partial t^2} - c^2\nabla^2 E = f(t, \mathbf{x}), \qquad (t, \mathbf{x}) \in [0, T[\times \Omega, \tag{1a}$$

$$\frac{\partial E}{\partial \hat{\mathbf{v}}} + hE = g(t, \mathbf{x}), \qquad (t, \mathbf{x}) \in [0, T[\times \Gamma, \tag{1b}$$

$$E(0, \mathbf{x}) = E^0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1c}$$

$$\frac{\partial E}{\partial t}(0, \mathbf{x}) = V^0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1d}$$

where $\mathbf{x} = (x, y)^\top$ are the Cartesian coordinates, t is the time variable, $\hat{\mathbf{v}}$ the outward unit normal on Γ, c and h are constants, and E the magnitude of the transverse electric field in the direction perpendicular to the domain plane. In (1), $f(t, \mathbf{x})$ and $g(t, \mathbf{x})$ are respectively, prescribed source and boundary functions, $E^0(\mathbf{x})$ and $V^0(\mathbf{x})$ are given initial conditions. Note that the model (1) represents the basis of many linear electromagnetic and acoustic propagation problems. For instance, applied to separate components of the linear electromagnetic field, it can represent an accurate and efficient solution for a short pulse propagating over long distances.

The time integration of the system (1) can be carried out using any implicit scheme including Newmark methods to avoid the very small time steps that may be required in simulations for explicit time integration schemes. However, the proposed spatial enrichment is time independent and hence independent of the choice of the integration scheme. The spatial discretization is introduced after the temporal one to enable changing the temporal discretization independently on the enrichment approach presented here. Alternative integration schemes can also be found in [2,11]. For simplicity in the presentation we consider the second-order central difference method. The latter is well known and details could be found in standard text books [1]. Thus, to integrate the Eqs. (1) in time we divide the time interval into N subintervals $[t_n, t_{n+1}]$ with length $\Delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots$. We use the notation W^n to denote the value of a generic function W at time t_n. Thus, given the solutions E^{n-1} and E^n at times t_{n-1} and t_n the solution at the next time step t_{n+1} is updated according to the semi-discrete equation

$$\begin{gathered}\frac{E^{n+1} - 2E^n + E^{n-1}}{\Delta t^2} - c^2\nabla^2 E^{n+1} = f(t_{n+1}, \mathbf{x}), \quad n = 0, 1, 2, \ldots, \\ E^0(\mathbf{x}) = E^0(\mathbf{x}), \\ E^{-1}(\mathbf{x}) = E^0(\mathbf{x}) - \Delta t V^0(\mathbf{x}).\end{gathered} \tag{2}$$

Note that to update the solution E^{n+1} in the semi-discrete formulation (2) one has to solve a linear system at each time step. The structure of this linear system is mainly dependent on the mesh used in the spatial discretization and the time step used in the time integration. For the spatial discretization, we multiply the equation in (2) by a weighting function $\phi(\mathbf{x})$, and then integrate over Ω. Using the divergence theorem and using the boundary condition (1b) one obtains the following weak formulation of the problem (1)

$$\begin{aligned}&\int_\Omega E^{n+1}\phi\, d\Omega + (c^2\Delta t^2)\int_\Omega \nabla E^{n+1}\cdot\nabla\phi\, d\Omega + (c^2\Delta t^2)\oint_\Gamma (hE^{n+1})\phi\, d\Gamma \\ &= \int_\Omega \left(2E^n - E^{n-1} + (\Delta t^2) f(t_{n+1}, \mathbf{x})\right)\phi\, d\Omega + (c^2\Delta t^2)\oint_\Gamma g(t_{n+1}, \mathbf{x})\phi\, d\Gamma.\end{aligned} \tag{3}$$

To solve the weak formulation (3) with the finite element method we discretize the spatial domain Ω into a set of finite elements $\mathcal{T}_i$ with the index i referring to the i-th element. The combination of all these elements forms our computational domain $\Omega_h = \cup_i \mathcal{T}_i$, with $\Omega_h \subseteq \Omega$. If $\mathcal{T}_i$ and $\mathcal{T}_j$ are two different elements of Ω_h, then $\mathcal{T}_i \cap \mathcal{T}_j$ is either a mesh point, or a common side, or the empty set. The