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Adaptive reduced basis strategy dedicated to the solution of nonstationary stochastic thermal problems

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ABSTRACT

This contribution addresses the modeling and the stochastic analysis of transient thermal processes by means of the finite element method. It focuses on the theoretical presentation as well as the application of an efficient reduced basis strategy that advantageously lowers the dimension of the investigated system. The modal content of the reduced basis is driven by the goal oriented error assessment of a user-defined quantity of interest. The first section of the article presents the stochastic system of interest: key aspects of a stochastic analysis are recalled along with the employed spatial discretization. The newly developed adaptive reduced basis strategy is then detailed in the second section before extensive numerical investigations are carried out in order to validate it in the last section of the article. A numerical benchmark allowing for the confrontation of the proposed strategy with usual Monte-Carlo simulations highlights the benefits of the method that allows for a precise control of the maximum admissible error on the quantity of interest.

1. Introduction

Ever-growing computational power allows to simulate physical phenomena of increasing complexity. In particular, the use of fine spatial discretization for sophisticated multiphysics numerical simulations is now conceivable for deterministic systems. For stochastic systems however, obtaining relevant results from numerical simulations is still a challenge since a very large number of simulations must be carried out. The stochastic nature of a system may be related to uncertainties inherent to its manufacturing [1,2] or to the fact that its solicitations—such as potential thermal gradients or pressure loads—are unknown.

There are two ways to model a stochastic problem: (1) one may consider a nonparametric probabilistic approach where the link between the values of the parameters and the mechanical model is not explicit as the one presented in [3] or (2) a parametric approach where the variability of design parameters is accounted for by means of parametric uncertainties may be employed. The method presented in this paper belongs to the second category.

In any case, a stochastic system features a randomness that can only be accounted for with a statistically relevant sample thus implying a very significant increase in terms of computational costs. A state of the art of stochastic numerical methods can be found in [4–6] and references therein. These methods, most of which rely on the well-known finite element method (FEM), may be split in three categories:

intrusive stochastic techniques are a generalization of the FEM accounting for uncertainties associated with the parameters of the problem. Both the variation of usual deterministic variables—space and time coordinates—and the random stochastic variables are discretized using the standard approach in the FEM: the Galerkin formulation [7–10]. The cornerstone of these methods lies in a proper definition of the approximation space of the stochastic variables. As a downside, these methods are computationally expensive and their implementation may be arduous. In order to overcome the high computational cost of these techniques, specific developments are available in the literature such as (1) iterative methods well-matched to the structure of resulting matrices [11–13] and (2) the use of reduced bases in order to represent the random space [9,14,15];

non-intrusive stochastic techniques are widely used as they rely on typical deterministic computations. Indeed, the randomness of the system is accounted for by means of Monte Carlo simulations. A large number of deterministic problems are solved through out the random space [16]. Such an approach is conceptually simple, robust and easy to implement but requires solving a

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large number of finite element problems in order to generate an output sample that is statistically relevant. The increased computation time inherent to such methods may be mitigated by the use of a reduced basis. In [17–19], the authors introduced a reduced basis methodology to reduce the cost of Monte Carlo simulations, offering an attractive framework for solving stochastic problems with a large number of parameters. The idea is simple and effective because the different Monte Carlo shots lead to similar FE problems and therefore the reduced basis approach is highly performant.

Uncertainty quantification techniques may introduce further an estimation of the analysis. Different methods have been proposed to control the approximations, see for example [20–23].

The method presented in this article can be classified as a non-intrusive technique. Indeed, it does not require any modification of the finite element formulation. However, and contrary to a classical Monte-Carlo approach, it does impose a very specific solution algorithm. It builds up on the idea of using a reduced basis combined with a goal oriented error assessment criterion which controls the content of the reduced basis in order to maximize its numerical efficacy. The present investigation is an extension of the work initiated in [24,25] to transient thermal processes [26,27]. As a matter of fact, the time dependence of the investigated problem makes it more arduous to solve. Indeed, the methodology introduced in [24] is specific to stationnary problems: assessing its applicability and relevance in the case of nonstationnary problems is the focus of this article.

In the first section of this article, the investigated thermal problem is presented in details: transient thermal equations are given before a brief description on how to tackle a stochastic problem with the usual Monte-Carlo strategy. In the second section, the proposed numerical strategy based on adaptive reduced basis is exposed. Theoretical details are given with respect to the problem formulation, the general algorithm and the error assessment procedure. In the last section of the article, numerical results are presented for a benchmark test: both the accuracy and the efficacy of the proposed numerical strategy are thoroughly analyzed.

2. Stochastic problem

2.1. Governing equations

We consider a bounded domain Ω, representing a 2D structure. The boundary $\partial\Omega$ of Ω, is divided in two parts $\partial_D\Omega$ and $\partial_N\Omega$ such that $\partial_D\Omega \cup \partial_N\Omega = \partial\Omega$and $\partial_D\Omega \cap \partial_N\Omega = \varnothing$.

The employed material model is assumed to be isotropic with no temperature dependence. Relevant material parameters mentioned in the following include: the density ρ, the specific heat c and the thermal conductivity λ. The stochastic behavior of the model is introduced assuming that $\lambda(\mathbf{x},\theta)$ is a random field, where $\mathbf{x} \in \Omega$ stands for the position vector, and $\theta \in \Theta$ characterizes the randomness. The sample space Θ is the set of possible outcomes of θ. As a random field, $\lambda(\mathbf{x},\theta)$ is a function mapping each point vector $\mathbf{x}$ to a random variable, typically with all the same Probability Density Function (PDF) and with cross-correlation depending on the distance between the locations. Assuming that the spatial correlation is regular enough, the Karhunen-Loève decomposition [28,29] allows for a representation of a random field by a sum of independent scalar random variables multiplied by deterministic functions of $\mathbf{x}$.

A prescribed flux field $\mathbf{f}_d$ is applied on $\partial_N\Omega$, a prescribed temperature field T_d is imposed on $\partial_D\Omega$ and a prescribed source field $\mathbf{r}_d$ is applied on Ω. In a general context, the material properties characterized by ρ, c and λ as well as the loadings $\mathbf{f}_d, T_d$ and r_d may be random fields. Without loss of generality, it is assumed in the following that the randomness is restricted to the material parameters introduced in the thermal conductivity λ. Thus, the problem reads: find the unknown temperature field $T(\mathbf{x},t,\theta)$ such that

$$\begin{cases} \mathrm{div}[\lambda(\mathbf{x},\theta)\mathbf{grad}(T(\mathbf{x},t,\theta))] + r_d(\mathbf{x},\theta) = \rho(\mathbf{x},\theta)c(\mathbf{x},\theta)\frac{\partial T}{\partial t}(\mathbf{x},t,\theta) & \text{in } \Omega \quad \text{(a)} \\ \mathbf{grad}(T(\mathbf{x},t,\theta)).\mathbf{n} = \mathbf{f}_d(\mathbf{x},t,\theta) & \text{on } \partial_N\Omega \quad \text{(b)} \\ T(\mathbf{x},t,\theta) = T_d(\mathbf{x},t,\theta) & \text{on } \partial_D\Omega \quad \text{(c)} \\ T(\mathbf{x},t=0,\theta) = T_0(\mathbf{x},\theta) & \text{on } \Omega \quad \text{(d)} \end{cases} \tag{1}$$

In the remainder, $\mathcal{T}$ refers to the set of admissible temperatures $T(\mathbf{x},t,\theta) \in \mathcal{T}$, satisfying (1c) and (1d).

2.2. Quantity of interest

The purpose of the stochastic analysis is to determine reliable statistical information of a response quantity of Interest I. Note that since the solution $T(\mathbf{x},t,\theta)$ is a random field, any output computed from this solution is a random quantity, and therefore the statistics of this output (expected value, variance...) are the relevant information to be estimated. In this article, the assumption is made that the quantity of interest may be expressed as a scalar quantity linearly dependent on $T(\mathbf{x},t,\theta)$: The purpose of the stochastic analysis is to determine reliable statistical information of a response quantity of Interest I. Note that since the solution $T(\mathbf{x},t,\theta)$ is a random field, any output computed from this solution is a random quantity, and therefore the statistics of this output (expected value, variance...) are the relevant information to be estimated. In this article, the assumption is made that the quantity of interest may be expressed as a scalar quantity linearly dependent on $T(\mathbf{x},t,\theta)$:

$$I(\theta) = \ell_I(T(\mathbf{x},t,\theta)), \tag{2}$$

where $\ell_I(\cdot)$ is a deterministic linear functional.

2.3. Reference solution

A reference solution is obtained using a non-intrusive approach that decouples the discretization of the physical space and the stochastic space, represented here by Ω and Θ. This can be described in two steps:

1. First, a few simplifications are introduced in order to solve the problem (1) and to obtain a numerical approximation of $T(\mathbf{x},t,\theta)$ for a realization of θ (freezing the randomness):
   - *Karhunen-Loéve truncation (Section 2.3.1):* the Karhunen-Loéve infinite expansion is approximated by limiting the sum to a finite number of terms, N_{KL},
   - *spatial discretization (Section 2.3.2):* the problem (1) is approximated as the application of the FEM yields a spatially discrete system,
   - *time discretization (Section 2.3.3)* is operated by means of the Crank-Nicholson time integration scheme.
2. Then, N_{MC} Monte Carlo simulations $\{\theta_k\}_{k=1,\dots,N_{\mathrm{MC}}}$ are used to obtain an approximation of the probability density function of the quantity of interest $I(\theta)$.

Details of each of the aforementioned steps are given hereafter.