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Strain smoothing for compressible and nearly-incompressible finite elasticity

Chang-Kye Lee [a], L. Angela Mihai [b], Jack S. Hale [c], Pierre Kerfriden [a], Stéphane P.A. Bordas [c,a,*]

[a] Cardiff School of Engineering, Cardiff University, The Queen's Building, The Parade, Cardiff, Wales CF24 3AA, UK
[b] Cardiff School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, Wales CF24 4AG, UK
[c] Université du Luxembourg, Faculté des Sciences, de la Technologies et de la Communication, Campus Kirchberg, 6, rue Coudenhove-Kalergi, L-1359, Luxembourg

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ABSTRACT

We present a robust and efficient form of the smoothed finite element method (S-FEM) to simulate hyperelastic bodies with compressible and nearly-incompressible neo-Hookean behaviour. The resulting method is stable, free from volumetric locking and robust on highly distorted meshes. To ensure inf-sup stability of our method we add a cubic bubble function to each element. The weak form for the smoothed hyperelastic problem is derived analogously to that of smoothed linear elastic problem. Smoothed strains and smoothed deformation gradients are evaluated on sub-domains selected by either edge information (edge-based S-FEM, ES-FEM) or nodal information (node-based S-FEM, NS-FEM). Numerical examples are shown that demonstrate the efficiency and reliability of the proposed approach in the nearly-incompressible limit and on highly distorted meshes. We conclude that, strain smoothing is at least as accurate and stable, as the MINI element, for an equivalent problem size.

1. Introduction

Low-order simplex (triangular or tetrahedral) finite element methods (FEM) are widely used because of computational efficiency, simplicity of implementation and the availability of largely automatic mesh generation for complex geometries. However, the accuracy of the low-order simplex FEM suffers in the incompressible limit, an issue commonly referred to as volumetric locking, and also when the mesh becomes highly distorted.

To deal with these difficulties various numerical techniques have been developed. A classical approach is to use hexahedral elements instead of tetrahedral elements due to their superior performance in plasticity, nearly-incompressible and bending problems, and additionally their reduced sensitivity to highly distorted meshes. However, automatically generating high-quality conforming hexahedral meshes of complex geometries is still not possible, and for this reason it is desirable to develop improved methods that can use simplex meshes. Significant progress has, however, been done in this direction [1].

Another option is to move to higher-order polynomial simplex elements. While they are significantly better than linear tetrahedral elements in terms of accuracy this is at the expense of increased implementational and computational complexity, and sensitivity to distortion.

Nodally averaged simplex elements [2,3] can effectively deal with nearly-incompressible materials, but they still suffer from an overly stiff behaviour in certain cases [4].

Meshfree (or meshless) methods [5–7] are another option because of their improved accuracy on highly-distorted nodal layouts, but the locking problem is still a challenging issue that needs careful consideration [8]. To improve the non-mesh based methods, B-bar approach [9,10], which is appropriate not only to handle incompressible limits but also to model shear bands with cohesive surfaces, can be considered. Additionally, because they are substantially different to the FEM, they are not easily implemented in it existing software.

Isogeometric Analysis (IGA) is another high-order alternative and the interested reader is referred to [11,12]. Moreover for the further studies for fractures undergoing large deformations, edge rotation algorithm can be an another option in large plastic strains [13,14].

Mixed and enhanced formulations are another popular remedy for volumetric locking [15,16], but they retain the sensitivity to mesh distortion of the standard simplex FEM [17].

Another approach, and the one that we employ in this paper, is the strain smoothing method developed by Liu et al. [18,19]. The strain smoothing method has the advantage over the above methods that it improves both the behaviour of low-order simplex ele-

* Corresponding author at: Université du Luxembourg, Faculté des Sciences, de la Technologies et de la Communication, Campus Kirchberg, 6, rue Coudenhove-Kalergi, L-1359, Luxembourg.

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ments with respect to both volumetric locking and highly distorted meshes, while being simple to implement within an existing finite element code.

The basic idea of strain smoothing is based on the stabilised conforming nodal integration (SCNI) proposed in the context of meshfree methods by Chen et al. [20,21]. Later SCNI was extended to the natural element method (NEM) by Yoo et al. [22], and was shown to effectively handle nearly-incompressible problems.

In the smoothed finite element method (S-FEM), the domain is divided into smoothing domains where the strain is smoothed as shown in Fig. 1. Typically, the geometry of the smoothing domains is derived directly from the standard simplex mesh geometry. Then with the divergence theorem, numerical integration is transferred from the interior to the boundary of the smoothing domains [23,24]. Critically, this procedure results in a discrete weak form without the Jacobian, the matrix used to map basis function derivatives from the reference element to the real element in the mesh. In the standard FEM the Jacobian is required to construct the derivatives of the basis functions. When distorted meshes are used in the standard FEM, the Jacobian becomes ill-conditioned, and this affects the accuracy of the method. Because the Jacobian is not required in S-FEM, the resulting method is significantly more robust than the standard FEM on highly distorted meshes.

It is also known that the S-FEM produces stiffness matrices that are less stiff than the standard FEM, and in certain cases this property can be used to overcome volumetric locking. Since S-FEM was introduced, its properties have been studied from a theoretical viewpoint [18,19,25–29], extended to n-sided polygonal elements [30] and applied to many engineering problems such as plate and shell analysis [31–34].

Particularly, Bordas et al. [35] recalled the central theory and features of S-FEM and showed notable properties of S-FEM which depend on the number of smoothing domains in an element. Moreover, Bordas et al. [35] presented the coupling of strain smoothing and partition of unity enrichment, so called SmXFEM, with examples of cracks in linear elastic continua and arbitrary cracks in plates.

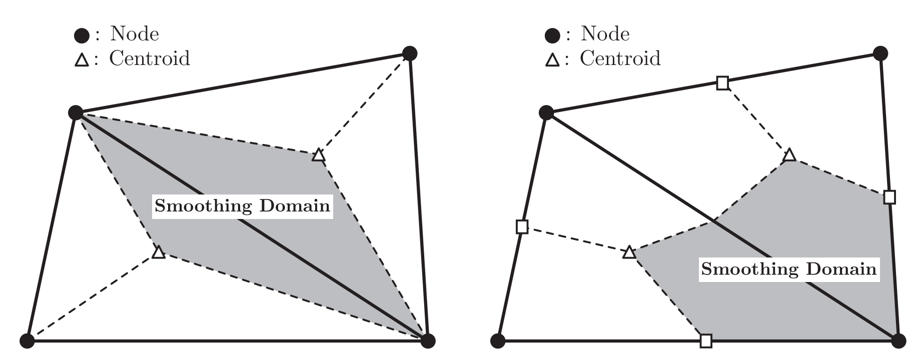
The contribution of this paper to the literature is to present a robust, efficient and stable form of the smoothed finite element methods to simulate both compressible and nearly-compressible hyperelastic bodies. We study two forms of smoothing (node-based and edge-based) and compare their relative merits. A key ingredient of our method is to add cubic bubbles to each element to ensure inf-sup stability. Although bubbles have been suggested before in the context of linear elastic S-FEM by Nguyen-Xuan and Liu [36] here we make the non-trivial extension to deal with hyperelastic problems. Finally we present a rigorous testing procedure that demonstrates the superior performance of our approach over the standard FEM.

The outline of this paper is as follows; first, we briefly review the idea fundamentals of S-FEM. In Section 3 we formulate the non-linear S-FEM for hyperelastic neo-Hookean compressible materials. To demonstrate the accuracy and convergence properties of the proposed methods we present extensive benchmark tests in Section 4. Finally, conclusions and future work directions are summarised in Section 5.

2. Smoothed finite element method (S-FEM)

It was shown in numerous studies that S-FEM provides a higher efficiency, i.e. computational cost versus error than the conventional FEM for many mechanical problems. We list below some of the strengths and weaknesses of each variant: the cell-based smoothed FEM (CS-FEM), the edge-based smoothed FEM (ES-FEM), the node-based smoothed FEM (NS-FEM), and the face-based smoothed FEM (FS-FEM).

- **Volumetric locking**. NS-FEM can handle effectively nearly-incompressible materials where Poisson's ratio $\nu \to 0.5$ [37], while ES-FEM suffers from volumetric locking. Combining NS- and ES-FEM gives the so-called *the smoothing-domain-based selective ES/NS-FEM* which also overcomes volumetric locking [38]. In the case of CS-FEM, volumetric locking can be avoided by selective integration [39].
- **Upper and lower bound properties**. In typical engineering analysis with homogeneous Dirichlet boundary conditions the NS-FEM gives upper bound solution and FEM obtains lower bound solution in the energy norm. While, in the case of problem with no external force but non-homogeneous Dirichlet boundary conditions, NS-FEM and FEM provide lower and upper bounds in the energy norm, respectively [40,41].
- **Static and dynamic analyses**. ES-FEM gives accurate and stable results when solving either static or dynamic problems [42]. In contrast, although NS-FEM is spatially stable, it is temporally unstable. Therefore, to solve dynamic problems, NS-FEM requires stabilisation techniques [43,44]. CS-FEM can also be extended to solve dynamic problems [45].
- **Other features**. In NS-FEM, the accuracy of the solution in the displacement norm is comparable to that of the standard FEM using the same mesh, whereas the accuracy of stress solutions



(a) The smoothing domain for edge-based smoothed FEM (ES-FEM)
(b) The smoothing domain for node-based smoothed FEM (NS-FEM)

Fig. 1. (a) Three smoothing domains in the three-node triangular (T3) finite mesh for edge-based smoothed FEM (ES-FEM), (b) three smoothing domains in the three-node triangular (T3) finite mesh for node-based smoothed FEM (NS-FEM).