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A novel ‘boundary layer’ finite element for the efficient analysis of thin cylindrical shells

Adrien Boyez, Adam J. Sadowski *, Bassam A. Izzuddin

Department of Civil & Environmental Engineering, Imperial College London, United Kingdom

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ABSTRACT

Classical shell finite elements usually employ low-order polynomial shape functions to interpolate between nodal displacement and rotational degrees of freedom. Consequently, carefully-designed fine meshes are often required to accurately capture regions of high local curvature, such as at the ‘boundary layer’ of bending that occurs in cylindrical shells near a boundary or discontinuity. This significantly increases the computational cost of any analysis.

This paper is a ‘proof of concept’ illustration of a novel cylindrical axisymmetric shell element that is enriched with rigorously-derived transcendental shape functions to exactly capture the bending boundary layer. When complemented with simple polynomials to express the membrane displacements, a single boundary layer shell element is able to support very complex displacement and stress fields that are exact for distributed element loads of up to second order. A single element is usually sufficient per shell segment in a multi-strake shell.

The predictions of the novel element are compared against analytical solutions, a classical axisymmetric shell element with polynomial shape functions and the ABAQUS S4R shell element in three problems of increasing complexity and practical relevance. The element displays excellent numerical results with only a fraction of the total degrees of freedom and involves virtually no mesh design. The shell theory employed at present is kept deliberately simple for illustration purposes, though the formulation will be extended in future work.

1. Introduction

Membrane action is the preferred load-carrying mechanism for shells, enabling efficient and economical use of material. As membrane forces can be obtained easily through equilibrium alone and are valid throughout much of the shell, membrane theory often forms the basis of design. However, bending action must be considered to fully take into account the effect of kinematic boundary conditions and to identify the range of validity of membrane action [1,2]. Bending theory is significantly more complex mathematically, and even the very simplest linear axisymmetric variant requires the solution of a fourth-order non-homogeneous differential equation [3–6]. The high order of the governing equations belies a rich set of underlying physical behaviours, chief among them being the possibility of displacements and stress fields exhibiting rapid variations and high magnitudes near boundaries or discontinuities. This ‘boundary layer’ decays exponentially away from boundaries at a rate governed by the bending half-wavelength λ, settling on a particular integral corresponding to membrane action [2].

As analytical solutions cannot easily be obtained even for simple shell bending problems [2,6–10], the finite element method (FEM) is widely employed instead [11–15]. Numerous shell element formulations exist, all based on polynomial shape functions of varying order. Membrane action is very ‘smooth’ and easily captured, but convergence to the solution in the vicinity of a bending boundary layer requires careful local mesh refinement [2,15,16]. Multi-segment or multi-strake shells may exhibit several boundary layers, each requiring a locally-refined interpolation field and contributing greatly to the total number of degrees of freedom in the system. For this reason, symmetry is exploited wherever possible for computational efficiency, although even axisymmetric shells exhibit boundary layers.

2. Scope of the study

The central concept behind the present study is to formally distinguish between membrane and bending components of the displacement solution at the level of the interpolation field, and to

* Corresponding author.
E-mail address: a.sadowski@imperial.ac.uk (A.J. Sadowski).

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enrich the field through specialised bending shape functions derived rigorously from the governing differential equation. In this way the boundary layer is included natively within the finite element, leading to significant gains in accuracy and substantial economies in terms of total degrees of freedom, modelling effort and mesh design. The idea of enriching the interpolation field to account for specific local and global phenomena is not new and is the basis of the eXtended or General FEM (XFEM or GFEM) methods [17–20], but to the authors' knowledge it is the first time that such an approach has been applied to shell elements specifically to account for localised bending phenomena. The complexity is purposefully limited here to the very minimum required to demonstrate the validity of the approach: the proposed Cylindrical Shell Boundary Layer (CSBL) element currently supports linear stress analysis of axisymmetric loading on thin cylindrical shells, based on a simple Kirchhoff-Love shell bending theory [21,22]. However the use of a general constitutive relation enables the study of isotropic, uniformly orthotropic and meridionally-stiffened 'smeared' shells [22–24], making it an efficient tool for the axisymmetric bending stress analysis of multi-segment cylinders, silos, tanks and pressure vessels even in its present form. The performance of the linear CSBL element is illustrated on three example problems of increasing complexity, two of which relate directly to non-trivial practical axisymmetric design problems.

3. Axisymmetric bending theory for thin orthotropic cylindrical shells

The idea of using specialised shape functions to capture the boundary layer specifically in cylindrical shells stems directly from an analytical result in classical shell bending theory. Here, the mathematical distinction between the homogeneous and particular solutions of the governing differential equation corresponds directly to physical bending and membrane action respectively. The kinematic relations are kept linear in what follows, as even a simple axisymmetric thin-walled shell theory based on the Kirchhoff-Love assumptions [7,21] captures the mechanics of meridional bending together with its associated boundary layer. This has the additional benefit that the solutions for the normal w and meridional u displacements are decoupled, permitting the origin of the proposed shape functions to be illustrated clearly. However, the linear constitutive relations are generalised to allow for the study of both isotropic and uniformly orthotropic cylinders via the 'smeared' stiffness approach [23,24]. Lastly, as the transcendental bending shape functions of the proposed CSBL element are obtained directly from the analytical solution to the governing differential equation, some level of detail in presenting its derivation, however classical, is necessary here.

Under axisymmetric conditions, a cylindrical shell of radius r and thickness t may be subject to pressure loading normal p_n and meridionally tangential p_z to the midsurface (dimensions of [F L^{-2}]), as shown in Fig. 1. Axisymmetry of the loading, boundary conditions and geometry ensures that only five stress resultants act on the mid-surface of the thin shell: the meridional and circumferential membrane stress resultants n_z and n_θ ([F L^{-1}]), the bending moment stress resultants m_z and m_θ ([FL L^{-1}]), and the meridional transverse shear stress resultant q_z ([F L^{-1}]). There are no displacements or gradients in the circumferential direction.

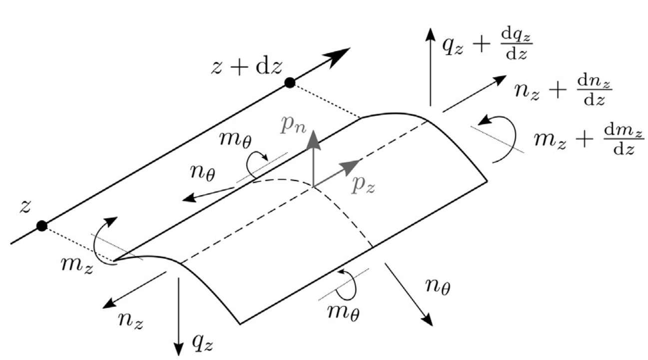

Fig. 1. Equilibrium of an element of a thin-walled axisymmetric cylindrical shell.

Considering equilibrium of an elementary cylinder section of length dz and arc length rdθ yields the following equations:

$$\frac{dn_z}{dz} = -p_z, \quad n_\theta = r\left(p_r + \frac{dq_z}{dz}\right) \quad \text{and } q_z = -\frac{dm_z}{dz} \tag{1}$$

The following constitutive and kinematic relations are used in this illustration [22]:

$$\begin{bmatrix} n_z \\ n_\theta \\ m_z \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & 0 \\ C_{13} & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_z \\ \varepsilon_\theta \\ \kappa_z \end{bmatrix} \tag{2}$$

$$\text{and } \begin{bmatrix} \varepsilon_z \\ \varepsilon_\theta \\ \kappa_z \end{bmatrix} = \begin{bmatrix} \frac{du}{dz} & \frac{w}{r} & \frac{d^2w}{dz^2} \end{bmatrix}^T \tag{3}$$

where the C's represent appropriate stiffness coefficients that will be discussed later. The resultants m_θ and q_z need not be included in Eq. (2) as their corresponding generalised strains are zero. Combining Eqs. (1)–(3) and simplifying the result leads to a linear fourth-order ordinary differential equation in w only, the normal midsurface displacement:

$$r\left(C_{11}C_{33} - C_{13}^2\right)\frac{d^4w}{dz^4} - 2C_{12}C_{13}\frac{d^2w}{dz^2} + \frac{1}{r}\left(C_{11}C_{22} - C_{12}^2\right)w$$
$$= rC_{11}p_r + C_{12}\left(\int_0^z p_z dz - n_{z0}\right) + rC_{13}\frac{dp_z}{dz} \tag{4}$$

Solving the homogeneous part of the equation requires finding the complex roots of the corresponding characteristic polynomial:

$$aX^4 + 2bX^2 + c = 0 \text{ where } \begin{cases} a = r\left(C_{11}C_{33} - C_{13}^2\right) \\ b = -C_{12}C_{13} \\ c = r^{-1}\left(C_{11}C_{22} - C_{12}^2\right) \end{cases} \tag{5}$$

Setting $Y = X^2$, this becomes a polynomial of second order in Y, for which the discriminant is:

$$\delta = b^2 - ac = C_{12}^2C_{13}^2 - (C_{11}C_{22} - C_{12}^2)(C_{11}C_{33} - C_{13}^2) \tag{6}$$

which is negative if and only if the following inequality is satisfied:

$$\frac{C_{12}^2}{C_{22}} + \frac{C_{13}^2}{C_{33}} < C_{11} \tag{7}$$

It is important to establish that this inequality will indeed always be satisfied, as this governs the functional form of the general solution to the homogeneous equation. For a very general uniformly orthotropic shell with elastic moduli E_z and E_θ, Poisson's ratio ν and thickness t, and 'smeared' meridional stiffeners of modulus E_s, cross-section area A_s, second moment of area I_s, spacing d_s and eccentricity e_s, the constitutive matrix [**C**] is the following [22]:

$$[\mathbf{C}] = \begin{bmatrix} \frac{E_zt}{1-\nu^2} + \frac{E_sA_s}{d_s} & \nu\frac{\sqrt{E_zE_\theta}t}{1-\nu^2} & \frac{e_sE_sA_s}{d_s} \\ \nu\frac{\sqrt{E_zE_\theta}t}{1-\nu^2} & \frac{E_\theta t}{1-\nu^2} & 0 \\ \frac{e_sE_sA_s}{d_s} & 0 & \frac{E_zt^3}{12(1-\nu^2)} + \frac{E_sI_s}{d_s} + \frac{e_s^2E_sA_s}{d_s} \end{bmatrix} \tag{8}$$

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