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A mixed formulation for nonlinear analysis of cable structures

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ABSTRACT

This paper proposes a general finite-element procedure for the nonlinear analysis of cables based on a mixed variational formulation in curvilinear coordinates with finite deformations. The formulation accounts for nonlinear elasticity and inelasticity, overcoming the limitation of recent numerical approaches which integrate explicitly the global balance of linear momentum for a linear elastic material with infinitesimal deformations. The formulation uses a weak form of the catenary problem and of the strain-displacement relation to derive a new family of cable finite elements with a continuous or discontinuous axial force field. Several examples from the literature on nonlinear cable analysis are used to validate the proposed formulation for St. Venant-Kirchhoff elastic materials and neo-Hookean materials. These studies show that the proposed formulation captures the displacements and the axial force distribution with high accuracy using a small number of finite elements.

1. Introduction

Cable structures are widely used in engineering practice because they offer the advantages of high ultimate strength, flexibility, light weight and prestressing capabilities, among others. Because the behavior of a flexible cable is highly nonlinear, significant effort has been invested into developing accurate and economical numerical models for it. These models have evolved from truss elements to elastic elements satisfying the catenary equation. The survey in Tibert [27] provides a detailed overview of the different models. The following brief comments highlight the main features of the two basic approaches and point out some of their limitations. The simplest approach involves the representation of the cable as a series of straight truss elements. These are often based on linear displacement interpolation functions in the context of infinitesimal-deformation theory [20,17,19]. The geometric nonlinearity is often accounted for with the corotational formulation [9], involving the transformation of the node kinematic variables under large displacements. These elements suffer from the excessive mesh refinement required to accurately capture the deformed shape and the axial force distribution, especially when using linear shape functions and thus a constant axial force in the element. Also, because these elements are not specifically formulated as cables, they may exhibit a snap-through instability at states of nearly singular stiffness.

To address the excessive mesh refinement limitation of simple truss elements, catenary elements have been proposed. These elements formulate the global balance of linear momentum assuming one-dimensional infinitesimal linear elasticity (Hooke's law) and obtain the deformed shape by explicit integration [3,29,24,1,2]. Such et al. [24] and Ahmad Abad et al. [1] also proposed a finite-difference version of this catenary formulation by discretizing the global balance of linear momentum into n segments.

While catenary formulations give more accurate results than truss elements for the same mesh discretization, they also have shortcomings that limit their range of application. First, current catenary elements do not support extension to finite deformations and nonlinear material behavior. Second, these elements assume infinitesimal deformations and integrate the global balance of linear momentum explicitly without distinction between the 2nd Piola-Kirchhoff (2nd PK) and Cauchy representations of the axial force [12, Ch. 9]. Third, this explicit integration does not accommodate a consistent mass matrix for dynamic analysis, limiting such approaches to the use of a lumped mass with the consequence that a large number of elements is required for accuracy [26]. Finally, because of the assumption of infinitesimal deformations, the distributed loads do not evolve consistently with the cable elongation, resulting in the inaccurate balance of linear momentum in the deformed configuration. To address this problem, *associated catenary elements* [3] impose restrictions of the form $wL = \tilde{w}l$ with w and L the load and length in the reference configuration, and $\tilde{w}$ and l the load and length in the current configuration, respectively.

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To address the limitations of truss and catenary cable elements, this study proposes a new formulation for a family of cable finite elements with the following objectives:

- Use finite deformation theory to describe the geometric nonlinearity.
- Solve the balance of linear momentum consistently.
- Accommodate nonlinear elastic material response.
- Can be extended to inelastic material response.
- Develop a robust and versatile finite element implementation to allow its deployment in a general purpose finite element analysis framework.

The presentation starts with the formulation of the cable kinematics under finite deformations in Section 2, and proceeds to derive the principle of virtual work and weak compatibility relation in Section 3. Subsequently, Section 4 presents the finite element implementation of the weak form of the governing equations, and discusses the numerical stability requirements of the formulation. Following a brief discussion of nonlinear elastic material models in Section 6, the presentation sets the stage for the subsequent numerical studies by describing first the solution of the form finding problem with the current formulation in Section 7. Finally, Section 8 assesses the accuracy and numerical convergence characteristics of the proposed elements with the study of cable problems from the literature.

2. Geometry and kinematics

2.1. Geometric preliminaries

Fig. 1 shows a curve $\mathcal{C}$ representing an idealized cable in three dimensions, with reference Cartesian coordinate system $\{\mathbf{E}_A\}_{A=1}^3$.

Define an orthogonal frame $\{\mathbf{G}_i\}_{i=1}^3$ at any material point $P \in \mathcal{C}$ associated with coordinates $\{\xi^i\}_{i=1}^3$ such that

$$\mathbf{G}_1 = \frac{d\mathbf{X}}{d\xi^1}; \quad \mathbf{G}_1 \cdot \mathbf{G}_2 = 0; \quad \|\mathbf{G}_2\| = 1; \quad \mathbf{G}_3 = \frac{\mathbf{G}_1 \times \mathbf{G}_2}{\|\mathbf{G}_1 \times \mathbf{G}_2\|} \tag{1}$$

where ξ^1 is the selected parameter for describing the curve. Note that the frame $\{\mathbf{G}_i\}_{i=1}^3$ is orthogonal but, in general, not orthonormal. Indeed, the metric tensor $[G_{ij}]$ for the frame $\{\mathbf{G}_i\}_{i=1}^3$ is of the form

$$[G_{ij}] = [\mathbf{G}_i \cdot \mathbf{G}_j] = \begin{bmatrix} \|\mathbf{G}_1\|^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

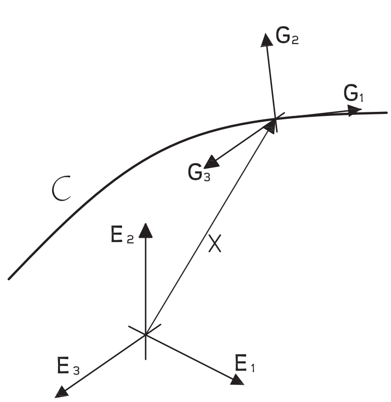

Fig. 1. Geometry for general cable.

Therefore, the differential vector $d\mathbf{X}$ along the curve $\mathcal{C}$ is

$$d\mathbf{X} = \frac{d\mathbf{X}}{d\xi^1} d\xi^1 = \mathbf{G}_1 d\xi^1 \tag{3}$$

The corresponding dual frame $\{\mathbf{G}^i\}_{i=1}^3$ satisfies

$$\mathbf{G}_i = G_{ij}\mathbf{G}^j \Rightarrow \mathbf{G}^i = G^{ij}\mathbf{G}_j = G_{ij}^{-1}\mathbf{G}_j \tag{4}$$

where $[G^{ij}]$ represents the dual metric tensor.

2.2. Finite-deformation kinematics

With the preceding definitions, let the cable $\mathcal{C}$ undergo the motion $\boldsymbol{\chi}(\xi^1) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ in Fig. 2, such that $\boldsymbol{\chi}(\mathbf{X}(\xi^1)) = \mathbf{x}$, where $\mathbf{X}$ represents the position vector in the reference configuration $\mathcal{P}_0$ and $\mathbf{x}$, the position vector in the current configuration $\mathcal{P}$. Upper case letters denote the variables in the reference configuration and lower case letters, the variables in the current configuration. Note that the cable motion can be completely described by the single coordinate ξ^1, because the cable is idealized as a one-dimensional manifold [6].

In the global Cartesian coordinate system, $\mathbf{X} = X_A\mathbf{E}_A$ defines the reference coordinates, whereas $\mathbf{x} = x_i\mathbf{e}_i$ defines the current coordinates. Note that, in Cartesian coordinates, $\mathbf{E}_A = \mathbf{E}^A$ and $\mathbf{e}_i = \mathbf{e}^i$. Let $\{\mathbf{E}_A\}_{A=1}^3 \equiv \{\mathbf{e}_i\}_{i=1}^3$, for simplicity.

Convecting the curvilinear coordinates $\{\xi^i\}_{i=1}^3$ with basis $\{\mathbf{G}_i\}_{i=1}^3$ into the coordinates $\{\eta^i\}_{i=1}^3$ with basis $\{\mathbf{g}_i\}_{i=1}^3$, the deformation gradient $\mathbf{F}$ and the right Cauchy-Green tensor $\mathbf{C}$ are [23]

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{g}_i \otimes \mathbf{G}^i; \quad \mathbf{C} = \mathbf{F}^t\mathbf{F} = g_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \tag{5}$$

Thus, $\mathbf{F}\mathbf{G}_1 = \mathbf{g}_1$ and $\mathbf{G}_1 = \mathbf{F}^{-1}\mathbf{g}_1$. The Green-Lagrange strain tensor $\mathbf{E}$ is

$$\mathbf{E} = \frac{1}{2}(g_{ij} - G_{ij})\mathbf{G}^i \otimes \mathbf{G}^j \tag{6}$$

Hence the only nonzero strain arises in the $\mathbf{G}_1$ direction. It is

$$\mathrm{E}_{11} = \frac{1}{2}(\|\mathbf{g}_1\|^2 - \|\mathbf{G}_1\|^2) \tag{7}$$

The relevant stretch λ of the problem in the $\mathbf{G}_1$ direction is

$$\lambda^2 = \left(\frac{ds}{dS}\right)^2 = \frac{d\mathbf{x} \cdot d\mathbf{x}}{d\mathbf{X} \cdot d\mathbf{X}} = \frac{g_{11}}{G_{11}} = \left(\frac{\|\mathbf{g}_1\|}{\|\mathbf{G}_1\|}\right)^2 \tag{8}$$

Consequently,

$$\mathrm{E}_{11} = \frac{1}{2}(\lambda^2 - 1)\|\mathbf{G}_1\|^2 \tag{9}$$

The displacement vector $\mathbf{u}$ depends only on the curvilinear coordinate ξ^1 of the material point $P \in \mathcal{C}$

$$\mathbf{u}(\mathbf{X}(\xi^1)) = \mathbf{x}(\mathbf{X}(\xi^1)) - \mathbf{X}(\xi^1) = u_A(\xi^1)\mathbf{E}_A \tag{10}$$

For the referential displacement gradient $\mathbf{H}$ in curvilinear coordinates, one observes that the only nonzero derivative with respect to $\{\xi^i\}_{i=1}^3$ is

$$\frac{d\mathbf{u}}{d\xi^1} = \frac{d\mathbf{x}}{d\xi^1} - \frac{d\mathbf{X}}{d\xi^1} = \mathbf{g}_1 - \mathbf{G}_1 \tag{11}$$

As a result,

$$\mathbf{H} = \frac{d\mathbf{u}}{d\xi^1} \otimes \mathbf{G}^1 \tag{12}$$

so that the relationship between the Green-Lagrange strain $\mathbf{E}$ and the displacement field $\mathbf{u}$ is