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Analysis of structures with random axial stiffness described by imprecise probability density functions

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This paper addresses the analysis of structures with random axial stiffness described by *imprecise probability density function* (*PDF*). Uncertainties are modelled as random variables whose *PDF* is assumed to depend on interval basic parameters (mean-value, variance, etc.). The main purpose of the analysis is to propagate the *imprecise PDF* of the random axial stiffness by establishing approximate bounds on the mean-value and variance of the response. To this aim, an efficient method is proposed which relies on the combination of standard probabilistic analysis with the so-called *improved interval analysis via extra unitary interval* and the *Rational Series Expansion*, recently introduced by the authors. The accuracy of the proposed bounds of response statistics is demonstrated by appropriate comparisons with the results obtained performing standard *Monte Carlo Simulation* in conjunction with a combinatorial procedure.

1. Introduction

Uncertainties affecting both structural parameters (e.g. material and/or geometric properties, fabrication details, etc.) and external loads play a crucial role in the prediction of structural behavior. In the last decades, several methods have been developed to analyze the effects of uncertain properties on structural response. Such methods require some mathematical description of uncertainties based on available empirical information. The most common description is based on probability theory which treats the uncertain parameters as random variables or random fields with assigned *probability density function* (*PDF*). However, available data are often quite limited and of poor quality as well as imprecise, diffuse, fluctuating, incomplete, fragmentary, vague or ambiguous. It follows that available data are often insufficient to empirically determine the *PDF* of an uncertain variable. As a consequence, the "basic" parameters (e.g. mean-value, variance, etc.) of the *PDF* are affected by uncertainties. These uncertainties can sometimes be substantial and in many applications "precise probabilities" cannot be considered as adequate and credible models of real states. This issue has been the subject of considerable debate in the last decades and a new family of non-probabilistic or "possibilistic" assessment methods has been derived [1,2].

When the information relating to an uncertain quantity of interest is expressed only as a set of possible values that the quantity might take, this information is usually referred to as "imprecise". This is distinct from the conventional probabilistic treatment of uncertainty where a probability measure is assigned to possible values of the uncertain quantity. The extension of probabilistic analysis to include imprecise information is now well established in the theory of "imprecise probabilities" which may be viewed as a generalization of the traditional probability theory (see e.g., [3–6]). An imprecise probability arises when the probability for an event is bounded by a lower value and an upper value of the probability for the same event [5]. A key feature of imprecise probabilities is the identification of bounds on probabilities for events of interest; the uncertainty of an event is characterized by two measure values—a lower probability and an upper probability. The distance between the probability bounds reflects the indeterminacy in model specifications expressed as imprecision of the models. This imprecision is the concession for not introducing artificial model assumptions [2,7].

Different representations of imprecise probabilities have been proposed in the literature. For example, Dempster [3] and Shafer [4] formulated a theory, sometimes called *evidence theory*, which can be considered as a variant of probability theory, in which the elements of the sample space are not single points but sets of

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values. Walley [5] coined the term *imprecise probability*; his theory is based on the subjective behavioral interpretation of the probability with the lower and the upper previsions. Weichselberger [6] introduced the *interval probability* as a generalization of Kolmogorov's classical probability; the resulting theory does not depend upon interpretations of the probability concept. The generalization is performed through the use of lower and upper probabilities, denoted by $\underline{P}(A)$ and $\overline{P}(A)$, respectively, with $0 \leqslant \underline{P}(A) \leqslant \overline{P}(A) \leqslant 1$. The special case with $\underline{P}(A) = \overline{P}(A)$ for all events A provides precise probability, while $\underline{P}(A) = 0$ and $\overline{P}(A) = 1$ represents complete lack of knowledge about A. In order to unify the standard interval analysis [8–11] with the traditional probability theory, the *probability bounds analysis* was introduced [12,13]. In this approach, also known as *P-box*, an imprecise random variable is represented by upper and lower bounds of its *cumulative density function* (*CDF*), rather than upper and lower bounds of its *PDF*. The *fuzzy probability* has been formulated [14] considering probability distributions with fuzzy parameters. In the framework of imprecise probability, the structural reliability bounds have been afterwards determined by considering imprecise parameters of the *PDF* associated with the stress and strength [15,16]. Although being very general, the application of the previously described theories is often limited to simple models, mainly because of the computational burden associated to the propagation of the imprecise probability description [17,18].

In the framework of imprecise probability, the interval analysis is certainly a very effective tool for the evaluation of the bounds of response statistical moments. However, the application of approaches based on the *classical interval analysis* (*CIA*) to engineering problems is hindered by the so-called *dependency phenomenon* [2,19] which often leads to an overestimation of the interval result unacceptable for design purposes.

In this paper, a method for the analysis of structural systems with random axial stiffness is presented. Uncertainties are modelled as random variables with *imprecise PDF* so as to take into account that the *PDF* itself is subject to doubt. Specifically, the *imprecise PDF* is assumed to depend on interval "basic" parameters (e.g. means, variances, etc.) which possess bounded descriptions. The aim of the analysis is to propagate the *imprecise PDFs* of the random axial stiffness by establishing approximate bounds on the mean-value and variance of the response. The proposed approach relies on a combination of probabilistic and non-probabilistic tools. Specifically, the random character of uncertainty is handled by performing a standard probabilistic analysis while imprecision is processed by applying the *improved interval analysis via extra unitary interval* (*IIA via EUI*) [20] in conjunction with the so-called *Rational Series Expansion* (*RSE*) (see e.g., [21–23]). The main steps of the proposed procedure may be summarized as follows: (i) the derivation of approximate analytical expressions of the interval mean-value and variance of structural response by applying the *RSE* which enables to determine the inverse of the random stiffness matrix in approximate explicit form; (ii) the evaluation of explicit bounds of the interval mean-value and variance of structural response by adopting the *IIA via EUI* as an effective remedy to the overestimation due to the *dependency phenomenon*.

To demonstrate the effectiveness of the presented procedure, a braced shear-type frame and a 3D truss structure with random axial stiffness of braces and bars, respectively, characterized by *imprecise PDF* are analyzed.

The paper is organized as follows: in Section 2, preliminary concepts and definitions concerning the *imprecise PDF* model assumed in the paper are introduced; in Section 3, approximate explicit expressions of the mean-value and variance of displacements of structures with random axial stiffness are derived by means of the *RSE*; in Section 4, under the assumption of *imprecise PDF* of the random axial stiffness, approximate explicit expressions of the bounds of the interval mean-value and variance of displacements are derived; finally, in Section 5, numerical results are presented to demonstrate the accuracy and efficiency of the proposed method.

2. Imprecise probability: Preliminary concepts and definitions

Information on an uncertain quantity of interest is usually referred to as "imprecise" when it is expressed only as a set of possible values that the quantity might take. Here, in the framework of imprecise probability analysis, it is assumed that a random variable possesses a family of *probability density functions* (*PDFs*). In particular, let us introduce the function $p_X(x;\mathbf{a})$ which represents the family of *PDFs* of the random variable X (with $x \in \mathbb{R}$). This function, herein referred to as *imprecise PDF*, depends on the set of epistemic "basic" parameters $a_1, a_2, \ldots, a_s$, collected into the vector $\mathbf{a} = [a_1, a_2, \ldots, a_s]^{\mathrm{T}}$, that lies within the admissible closed region Q. Hereafter, it is assumed that the epistemic parameters define a bounded set of interval variables. This means that the vector $\mathbf{a}$ is constrained by an s-dimensional box Q. According to the *interval analysis* formalism, the set-interval vector of epistemic parameters $\mathbf{a}$ is represented by $\mathbf{a}^I \triangleq [\underline{\mathbf{a}}, \overline{\mathbf{a}}] \in \mathbb{IR}^s$, such that $\underline{\mathbf{a}} \leqslant \mathbf{a} \leqslant \overline{\mathbf{a}}$, where $\mathbb{IR}$ is the set of all closed real interval numbers. The symbols $\underline{\mathbf{a}}$ and $\overline{\mathbf{a}}$ denote the lower bound (LB) and upper bound (UB) vectors, while the apex I characterizes interval variables; the i-th element of the interval vector $\mathbf{a}^I$ can be defined as $a_i^I \triangleq [\underline{a}_i, \overline{a}_i]$, where $a_i^I \in \mathbb{IR}$, $\underline{a}_i$ and $\overline{a}_i$ are the LB and UB of the i-th epistemic basic parameter a_i^I, respectively.

In order to limit the overestimation due to the *dependency phenomenon* [2,19], interval computations are performed by applying the so-called *improved interval analysis via extra unitary interval* (*IIA via EUI*) [20]. Accordingly, the interval basic parameters a_i^I are expressed in the following *affine form*:

$$a_i^I = a_{0,i} + \Delta a_i \hat{e}_i^I \tag{1}$$

where

$$a_{0,i} = \frac{\underline{a}_i + \overline{a}_i}{2}; \qquad \Delta a_i = \frac{\overline{a}_i - \underline{a}_i}{2} \tag{2a, b}$$

are the midpoint value and deviation amplitude, while $\hat{e}_i^I = [-1, 1]$ is the *EUI* associated to the i-th interval basic parameter.

As can be readily inferred, the statistical moment of order k of the random variable X with *imprecise PDF* $p_X(x;\mathbf{a}^I)$ is defined by an interval. Indeed, the set of *PDFs* describing the random variable X yields a set of statistical moments. This concept is formally expressed by introducing the so-called *interval stochastic average operator* $\mathrm{E}^I\langle\bullet\rangle$, i.e.:

$$\mathrm{E}^I\langle X^k\rangle = \left[\underline{\mathrm{E}}\langle X^k\rangle, \overline{\mathrm{E}}\langle X^k\rangle\right]. \tag{3}$$

In the previous expression, $\underline{\mathrm{E}}\langle X^k\rangle$ and $\overline{\mathrm{E}}\langle X^k\rangle$ denote the LB and UB, respectively, of the k-th order statistical moment of the random variable X, characterized by the *imprecise PDF* $p_X(x;\mathbf{a}^I)$. Based on standard probability theory and *classical interval analysis* (*CIA*), such bounds can be evaluated as:

$$\underline{\mathrm{E}}\langle X^k\rangle = \min_{\mathbf{a}^I \in Q}\left\{\int_{-\infty}^{\infty} x^k\, p_X(x;\mathbf{a}^I)\,\mathrm{d}x\right\}; \qquad \overline{\mathrm{E}}\langle X^k\rangle = \max_{\mathbf{a}^I \in Q}\left\{\int_{-\infty}^{\infty} x^k\, p_X(x;\mathbf{a}^I)\,\mathrm{d}x\right\} \tag{4a, b}$$

where the symbols $\min\{\bullet\}$ and $\max\{\bullet\}$ mean minimum (inferior) and maximum (superior) value of the quantity into parentheses under the condition that $\mathbf{a}^I \in Q$, respectively.