Computers and Structures xxx (2016) xxx–xxx

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

A *p*-adaptation method for compressible flow problems using a goal-based error indicator

D. Ekelschot, D. Moxey, S.J. Sherwin, J. Peiró *

Department of Aeronautics, Imperial College London, London SW7 2AZ, UK

ARTICLE INFO

Article history:
Accepted 12 March 2016
Available online xxxx

Keywords:
High-order spectral/*hp* element method
Discontinuous Galerkin
Adjoint methods
p-adaptation
Goal-based error estimation

ABSTRACT

An accurate calculation of aerodynamic force coefficients for a given geometry is of fundamental importance for aircraft design. High-order spectral/*hp* element methods, which use a discontinuous Galerkin discretisation of the compressible Navier–Stokes equations, are now increasingly being used to improve the accuracy of flow simulations and thus the force coefficients. To reduce error in the calculated force coefficients whilst keeping computational cost minimal, we propose a *p*-adaptation method where the degree of the approximating polynomial is locally increased in the regions of the flow where low resolution is identified using a goal-based error estimator as follows.

Given an objective functional such as the aerodynamic force coefficients, we use control theory to derive an adjoint problem which provides the sensitivity of the functional with respect to changes in the flow variables, and assume that these changes are represented by the local truncation error. In its final form, the goal-based error indicator represents the effect of truncation error on the objective functional, suitably weighted by the adjoint solution. Both flow governing and adjoint equations are solved by the same high-order method, where we allow the degree of the polynomial within an element to vary across the mesh.

We initially calculate a steady-state solution to the governing equations using a low polynomial order and use the goal-based error indicator to identify parts of the computational domain that require improved solution accuracy which is achieved by increasing the approximation order. We demonstrate the cost-effectiveness of our method across a range of polynomial orders by considering a number of examples in two- and three-dimensions and in subsonic and transonic flow regimes. Reductions in both the number of degrees of freedom required to resolve the force coefficients to a given error, as well as the computational cost, are both observed in using the *p*-adaptive technique.

1. Introduction

A problem of significant interest to the aeronautics industry is the development of numerical methods that are capable of accurately determining the lift or drag coefficient of a given wing geometry, while keeping the computational cost as low as possible. The value of these coefficients is highly dependent on the surrounding flow properties, as well as the geometry under consideration. The key to obtaining accurate values for these coefficients therefore lies in determining the areas within the domain that have the greatest effect on the value of the lift or drag coefficient. In other words, determining the sensitivity of the lift or drag coefficients with respect to its surroundings tells us where the local accuracy of the solution should be enhanced. Increasing the resolution in these regions permits us to evaluate the quantity of interest more accurately and improve the efficiency of the simulation.

Goal-based error estimation is a technique that is based around this philosophy, providing an indication of the accuracy of a numerical solution that is based on a pre-defined target quantity of interest, such as the lift and drag coefficients. It relies on the concept of duality, in which an adjoint problem is derived from the governing equations. The solution to this adjoint problem represents the sensitivity to an infinitesimal perturbation on the target, and a local error indicator is defined as the inner product of the residual and the corresponding adjoint variable.

The resulting error indicator provides a way to adaptively increase computational resolution only in the regions of the domain where additional accuracy is needed, which keeps computational costs lower. There is a rich catalogue of literature available

* Corresponding author.
E-mail addresses: d.ekelschot12@imperial.ac.uk (D. Ekelschot), d.moxey@imperial.ac.uk (D. Moxey), s.sherwin@imperial.ac.uk (S.J. Sherwin), j.peiro@imperial.ac.uk (J. Peiró).

http://dx.doi.org/10.1016/j.compstruc.2016.03.004
0045-7949/

regarding the application of goal-based error estimation for mesh adaptation, which is discussed in the review by Fidkowski and Darmofal [1]. As a brief overview, the idea of using adjoint equations for goal-based error estimation with the finite element method has been outlined by Becker and Rannacher [2] and Larson and Barth [3]. Giles and Pierce [4] described how to make use of the concept of duality for optimising objective functionals for typical computational fluid dynamics problems such as lift and drag force coefficients. Applications of goal-based error estimation to compressible inviscid flow problems using a finite element discretisation is further described in Ref. [5,6]. Furthermore, goal-based adaptation for inviscid supersonic flow problems discretised using a finite volume discretisation was presented by Venditti and Darmofal [7].

However, the aforementioned works rely on h-refinement to drive the adaptation process, whereby elements within the mesh that represent the computational domain are subdivided, thereby reducing their size, increasing resolution and obtaining solutions of greater accuracy. More recently however, the use of high-order finite element methods, such as the spectral/hp element method, is becoming increasingly popular in the investigations of these aeronautics problems. These methods typically utilise high-order polynomial approximations on each element, as opposed to the traditional linear shape functions. They therefore possess a variety of properties that make them attractive in fluid research applications, including low numerical diffusion and dispersion characteristics, highly-scalable parallel implementations on modern hardware and the ability to obtain higher accuracy solutions at levels of computing time comparable to more traditional, lower order finite element and finite volume methods.

The use of high-order methods opens an alternative route to drive the adaptive process, which is the focus of this paper. Instead of subdividing elements, we may instead choose to vary the polynomial order, P, within a given element in order to either increase or decrease the computational resolution. Whilst high-order methods have been used in combination with goal-based error estimation and h-refinement, for example in Ref. [8], p-refinement has received far less attention, and work has classically focused on elliptic problems. Demkowicz et al. [9] proposed a fully automatic h- and p-adaptation strategy that was initially applied to elliptic problems where the projection based interpolation error of a fine reference solution was minimised. This was later on extended by incorporating a dual problem in the work of Solín and Demkowicz [10]. The potential benefits of using p-adaptation for compressible flows has been discussed by Li and Jameson [11], who compare h- and p-adaptation for external flow problems in the framework of the spectral differences. They found that p-adaptation provides the highest accuracy with respect to the number of degrees of freedom and CPU time. Recently, Giorgiani et al. [12] studied the propagation of waves in a harbour for which they proposed a p-adaptive hybridisable discontinuous Galerkin (HDG) approach. They were able to show that this approach outperforms the classical continuous Galerkin approach. However, solutions with shocks require h-adaptation instead of p-adaptation to avoid numerical oscillations. More recently, the combination of h- and p-adaptation for compressible flows has been explored in Ref. [13].

The purpose of this work is to build on this body of knowledge and describe a novel scheme that can highlight the ability of p-adaptation, in combination with goal-based error estimation, to more accurately calculate aerodynamic force coefficients at a low computational cost. In particular, we present a high-order spectral/hp discontinuous Galerkin formulation of the compressible Euler and Navier–Stokes equations, which describes the underlying equations, adjoint problem and their implementation in detail. We will then consider a variety of examples of inviscid and viscous external flows in both two- and three-dimensions, considering both subsonic and transonic flow regimes, in order to examine the effectiveness of the p-adaptive method.

We conclude the introduction with a brief outline. First, the governing equations are introduced in Section 2. This is followed by a description of the goal-based error indicator in Section 3 and a derivation of the continuous adjoint equations for the compressible Navier–Stokes equations in Section 4. An outline of the high-order discretisation of the governing equations and corresponding adjoint equations is given in Section 5. Finally, the effectiveness of goal-based p-adaptation is assessed using a set of two- and three-dimensional numerical examples in Section 6. This set consist of three cases of two-dimensional flow past a NACA0012 aerofoil section: a subsonic inviscid flow ($Ma = 0.4$) at an incidence of five degrees, a subsonic laminar flow ($Ma = 0.1,\ Re = 5000$) at an angle of incidence of two degrees, and a transonic inviscid flow ($Ma = 0.8$) at an incidence of 1.25 degrees. The set is completed by a case involving the three-dimensional inviscid flow ($Ma = 0.5$) past an ellipsoid at an incidence of three degrees. Based on these results, we draw conclusions on the performance of the proposed p-adaptation strategy in Section 7.

2. Governing equations

We consider a compressible flow in which the physical laws of conservation of mass, momentum and energy for fluids in a domain, Ω, are described using the compressible Navier–Stokes equations

$$\boldsymbol{R}(\boldsymbol{u}, \nabla\boldsymbol{u}) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\{\boldsymbol{f}_i^c(\boldsymbol{u}) - \boldsymbol{f}_i^v(\boldsymbol{u}, \nabla\boldsymbol{u})\} = \boldsymbol{0}; \qquad \boldsymbol{u} \in \Omega \tag{1}$$

in a two-dimensional Cartesian frame of reference with coordinates (x_1, x_2). The vector of conserved variables is given by $\boldsymbol{u} = \{u_1, u_2, u_3, u_4\}^t = \{\rho, \rho v_1, \rho v_2, \rho E\}^t$ where ρ is the density, v_1 and v_2 are the Cartesian components of the velocity $\vec{v}$, and E is the total energy. Here $\boldsymbol{R}(\boldsymbol{u}, \nabla\boldsymbol{u})$ is used to denote the differential operator representing the governing equations with components $\boldsymbol{R} = \{R_1, R_2, R_3, R_4\}^t$. The Cartesian components of the convective fluxes, $\boldsymbol{f}_1^c$ and $\boldsymbol{f}_2^c$, are given by

$$\boldsymbol{f}_1^c = \begin{Bmatrix} \rho v_1 \\ p + \rho v_1^2 \\ \rho v_1 v_2 \\ \rho v_1 H \end{Bmatrix}, \quad \boldsymbol{f}_2^c = \begin{Bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ p + \rho v_2^2 \\ \rho v_2 H \end{Bmatrix} \tag{2}$$

where H is the total enthalpy and P is the pressure. The viscous fluxes are given by

$$\boldsymbol{f}_1^v = \begin{Bmatrix} 0 \\ \tau_{11} \\ \tau_{21} \\ v_1\tau_{11} + v_2\tau_{21} - k\frac{\partial T}{\partial x_1} \end{Bmatrix}, \quad \boldsymbol{f}_2^v = \begin{Bmatrix} 0 \\ \tau_{12} \\ \tau_{22} \\ v_1\tau_{12} + v_2\tau_{22} - k\frac{\partial T}{\partial x_2} \end{Bmatrix} \tag{3}$$

where T is the temperature, k is the thermal conductivity and τ is the tensor of viscous stresses, defined component-wise as

$$\begin{aligned} \tau_{11} &= \frac{4}{3}\mu\left(\frac{\partial v_1}{\partial x_1} - \frac{1}{2}\frac{\partial v_2}{\partial x_2}\right) \\ \tau_{12} &= \tau_{21} = \mu\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) \\ \tau_{22} &= \frac{4}{3}\mu\left(\frac{\partial v_2}{\partial x_2} - \frac{1}{2}\frac{\partial v_1}{\partial x_1}\right). \end{aligned} \tag{4}$$