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A high order local approximation free from linear dependency with quadrilateral mesh as mathematical cover and applications to linear elastic fractures

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ABSTRACT

The numerical manifold method falls into the category of the partition of unity methods. In order to enhance accuracy, high order polynomials can be specified as the local approximations. This, however, would incur rank deficiency of the stiffness matrix. In this study, a local displacement approximation is constructed over a physical patch generated from a four quadrilateral mathematical mesh. All the degrees of freedom are physically meaningful. The stresses are continuous at all nodes, suggesting that no stress polish is required. The proposed approximations have the same accuracy as the first-order polynomials, but no linear dependency inherent in the latter.

1. Introduction

Mechanical properties of the rock mass is determined by the rock blocks and various discontinuous structural planes. Many rock engineering practices have shown that rock mass failure usually starts from the propagation of internal discontinuity, then large deformation and large displacement follow, and finally engineering accidents happen. Thus it is of practical significance to study the whole process of fractured rock mass, including crack initiation, propagation and coalescence, sliding and finally forming the deposits. To this end, many numerical methods have been developed over the decades to solve the fracture problems.

Under the assumption of continuum, the finite element method (FEM) is the most commonly used in treating the discontinuous problems. There are mainly two models including the equivalent continuum model [1] and the joint or interface element model [2]. There still exist some disadvantages in the simulation of the crack problems with FEM: the finite element mesh must be in accordance with the crack; and remeshing is inevitable during the propagation of cracks.

In order to overcome the defects of FEM as mentioned above, the extended finite element method (XFEM) [3] and generalized finite element method (GFEM) [4] have been developed based on the partition of unity method (PUM). XFEM is an alternative to meshing or remeshing crack surfaces in computational fracture mechanics problems due to the concept of discontinuous and asymptotic partition of unity enrichment of the standard finite element approximation spaces [5]. In XFEM, the discontinuity of crack is simulated by introducing the generalized Heaviside functions; in addition, enrichment functions are also included to capture the stress singularity around crack tip more accurately. In principle, XFEM is not dependent on the finite mesh in tracking the crack, so it has been widely used in the crack growth problems [6–8]. But it still has difficulties in treating the large displacement problems. Recently, the strain smoothing technique in the smoothed FEM [9] (SFEM) proposed firstly by Liu is implanted into XFEM, which is not insensitive to mesh distortion and has a lower computational cost [10]. From then on, many successive excellent works have been done, such as the node-based smoothed XFEM (NS-XFEM) [11], extension of the strain smoothing technique to the higher order elements [12], edge-based XFEM (ESm-XFEM) [13] and combination of XFEM with the scaled boundary finite element method (SBFEM) [14]. They are all applied to solve the fracture problems and show good performance. In addition, an adaptive singular edge-based smoothed FEM (sES-FEM) [15] is a good improvement of the SFEM for the fracture problems. The newly developed isogeometric analysis (IGA) [16], which integrates the methods for analysis and Computer Aided Design (CAD) into a unified process, shows a great potential in solving the fracture problems.

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GFEM is nearly the same as the numerical manifold method (NMM) in essence except for the treatment of fractures and discrete blocks. The latter has been extended for application to rock mechanics problems with large deformation, whereas GFEM still has difficulties in simulating the movements of discrete rock block system [17]. Similarly, GFEM has been developed to simulate the three-dimensional dynamic crack propagation [18] and the branch crack problems [19].

Element-free method (EFM) is another continuum-based method in solving the strong discontinuity problems [20]. In EFM, the pre-processing is very easy even for those complex three-dimensional problems, because it only needs to discretize the problem domain by a group of nodes and the connection between nodes as in FEM is not necessary. The approximation functions can be directly constructed by the discrete nodes, so the mesh dependence is not as serious as in FEM. In treating the crack propagation problems, there are no mesh distortion and no need to remesh, which has greatly reduced the complexity. Similarly the enrichment functions as in XFEM can also be included to improve the accuracy of the stress field around the crack tip. EFM has been greatly extended to the three-dimensional fracture problems, such as a local partition of unity enriched element-free Galerkin method in which crack path continuity can be guaranteed [21], combination of the cohesive zone model [22], extended meshfree method without asymptotic enrichment where Lagrange multiplier field is added along the crack front to close the crack [23], the meshfree method based on the cracking-particle method [24] and new development of crack tracking procedure [25]. Furthermore, a detailed review of meshless methods based on the global weak forms in solid mechanics can be found in Ref. [26]. The shape functions in EFM are generally very complex, so the computation consummation is very large.

Discrete element method (DEM) and discontinuous deformation analysis (DDA) method are the two discontinuum-based methods in solving the fracture problems. DEM is firstly proposed by Cundall to study the mechanical behaviors of discontinuum such as rock mass [27]. DEM is an explicit algorithm which is based on Newton's second law. Rock mass is viewed as a series of rigid or deformable blocks cut by the discontinuities. The contact force model is represented by the tiny penetration between contact couples. DDA [28] proposed by Shi is an implicit method, which is based on the principle of minimum potential energy. Compared with DEM, DDA allows relatively large time steps and the stiffness matrix can be calculated by analytical simplex integration method. Both DEM and DDA allow large deformation, for example, Camones has utilized DEM to simulate crack propagation and coalescence [29]. Similarly, DDA has also been applied in predicting the failure process of the crack [30].

NMM proposed by Shi [31] can solve continuous and discontinuous problems of rock mechanics in a unified way. Recently it has been developed to solve the fourth-order problems [32]. In NMM, a mathematical patch might be cut into some physical patches, on which independent local approximations are defined. As a result, the discontinuity along a crack can be modeled more naturally. A lot of research work has been done, see Refs. [33–37].

It is no doubt that the high-order NMM with higher precision will be more suitable for the crack problems than the 0-order NMM. Here the high-order NMM refers to the first-order (or above) polynomials as the local approximations on the physical patches; while 0-order NMM polynomials means that constants are selected as the local approximations on the physical patches. However, the use of high-order polynomials is suffering from the linear dependence, where the global stiffness matrix is rank deficient even after the rigid body displacement modes are removed. The linear dependency issue is called as a 'nail' problem by its inventor. More details can be found in [38].

In this study, aiming at keeping the high precision and eliminating the linear dependency issue, a new displacement approximation scheme is proposed. Furthermore, the enrichment functions used to capture the singular stress field around crack tips are also included. Then the enhanced NMM is applied to elastic and fracture problems. The linear dependency issue has been resolved.

2. Foundation of numerical manifold method

NMM is based on the two cover systems including the mathematical cover (MC) and the physical cover (PC), so as to solve the continuous and discontinuous problems in a unified way. It should be pointed out that MC and PC are not independent from each other, PC is obtained by cutting MC with the components of the problem domain, including the boundary, the material interface and the discontinuity. Here, MC will be formed from a quadrilateral mathematical mesh.

An MC consists of a finite number of simply connected domains. Each domain is called as a mathematical patch (MP), which, in this study, is the union of several quadrilaterals sharing the same node such as MP-1 and MP-2 in Fig. 1. While deploying the MC, it is not necessary to force MC to be in accordance with the problem domain and it only needs to assure that the MC covers the problem domain completely.

PC is composed of all physical patches. The physical patches are generated by cutting all the mathematical patches, one by one, with the components of the problem domain. From one mathematical patch, therefore, more than one physical patch might be generated, such as PP-1, PP-2 and PP-*i* in Fig. 1.

Since physical patches partially overlap, a physical patch might be partitioned by other physical patch boundaries into disjointed domains. Each of these domains is referred to as a manifold element. As a result, a manifold element is a common domain of several physical patches. As shown in Fig. 1, the quadrilateral *i*-*j*-*m*-*l* with a segment of crack is a manifold element, which is the common region of physical patches PP-*i*, PP-*j*, PP-*m* and PP-*l*. Manifold elements are basic units in the numerical integration of the weak form of the problem.

In Fig. 1, there are two types of physical patches. Most physical patches are simply connected domains containing no crack tip, which are called nonsingular patches, such as PP-1. While a physical patch containing a crack tip is called as a singular patch, such as PP-*i*, in the center of Fig. 1. For different types of physical patches, different local approximations will be constructed as follows. Furthermore, the manifold elements are classified into three types: (1) normal manifold element covered only by nonsingular patches; (2) blending manifold element covered by both singular patches and nonsingular patches; (3) singular manifold element covered only by singular patches.

In addition, more details about NMM can be found in [34].

3. Construction of local approximations

In this section, a local approximation scheme based on the quadrilateral mathematical mesh is proposed by introducing new displacement approximations originating from the quadrilateral plate element [39] in FEM. The manifold element constructed in this way is denoted as Quad-P. The items of approximation functions and their properties are firstly presented. Then it is further extended to solve the linear elastic fracture problems.

3.1. Local displacement approximations on Quad-P

For the sake of completeness, a brief establishment of the Quad-P approximation functions is presented here. Let $\boldsymbol{x} = (x, y)$ be a point in