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Graph theoretical methods for efficient stochastic finite element analysis of structures

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ABSTRACT

Stochastic finite element method (StFEM) is a robust tool for uncertainty quantification of engineering systems having random properties. Nevertheless, the matrices involved in this method are very large compared to their deterministic counterparts. Thus, the computational aspects of StFEM are of great importance to be optimized. In this paper, an efficient StFEM is developed for analysis of structures. For this purpose, a method based on graph concepts is presented and extended to StFEM and recently developed stochastic spectral finite element method (StSFEM) procedures. Here, mathematical remedies are incorporated to enhance the analysis performance. Firstly, a graph theoretical method is presented for swift numerical solution of Fredholm integral equation arising from Karhunen–Loève expansion, which greatly reduces the existing computational cost, and can even be applied to the domain without symmetry. Secondly, a preconditioner is applied to decompose the matrices to Kronecker products of submatrices, and then graph product rules are utilized to solve the governing linear equation of cyclically symmetric models without inversing the final matrix, while only a small matrix is inversed instead. The proposed method provides significant improvement in the stochastic structural analysis. Illustrative examples demonstrate the efficiency and accuracy of the present method as a swift analysis.

1. Introduction

Mathematical modeling of systems can be either deterministic or stochastic. When stochastic modeling is used, one can incorporate uncertainties of an undertaken system, leading to a reliable design as uncertainty always exists in system properties, inherently. Uncertainties are categorized to *aleatoric* and *epistemic* uncertainties. Aleatoric uncertainty is an intrinsic variability of an especial quantity (e.g., the ocean wave force on an offshore structure), whereas *epistemic* uncertainty is due to lack of cognition about especial properties of a system, and unlike the aleatoric uncertainty (e.g., measurement of an underground fault), it can be decreased through data improvements.

Stochastic Finite Element Method (StFEM) is one of the most important numerical tools for uncertainty quantification in computational mechanics. Several types of StFEM exist in literature: perturbation method [1], Monte Carlo Simulation (MCS), StFEM with spectral decomposition [2] and other methods [3–5]. Development in efficient StFEM is still crucial and intriguing study, despite the fact that modern computers facilitate analysis of complex and large-scale problems. StFEM model of a problem needs much more computing system requirements in comparison to the deterministic FEM model of the same problem. There are many researches on improving and developing techniques for the analysis, among which some of them are referred here. Pellissetti and Ghanem [6] proposed an efficient iterative solvers for linear equations of StFEM. Panayirci et al. [7] implemented Guyan reduction for efficient stochastic analysis of structures by condensation of their deterministic model. Panayirci [8] solved Galerkin-based polynomial chaos expansion systems with preconditioned Conjugate Gradient (CG) solver and Cholesky decomposition. Matthies and Keese [9] implemented new algorithms for computing the mean and covariance of the solution of linear and nonlinear elliptic stochastic partial differential equations. Ullmann [10] proposed a Kronecker product preconditioner for StFEM reducing the number of iterations in preconditioned CG solver and also it was easily invertible. Sousedik and Ghanem [11,12] performed truncated hierarchical preconditioning for presentation of an effective StFEM solver. Khaji and Zakian [13] presented a stochastically enriched spectral finite element method (StSFEM) providing suitable accuracy and speed in dynamic analysis, diagonal mass matrix, minimal

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or sub-minimal mesh discretization. They [13] also proposed spectral finite element method (SFEM) for numerical solution of Karhunen–Loève Expansion (KLE). In an independent mathematical research, Oliveira and Azevedo [14] offered SFEM for eigensolution of Fredholm integral equation of second kind, simultaneously. Xu [15] accomplished quasi-weak and weak formulation of stochastic finite elements for static and dynamic problems. Chowdhury and Adhikari [16] represented a high dimensional model of StFEM for static and modal structural analyses. Lately, some studies have been concentrated on efficient stochastic modeling and reduction techniques for uncertainty computations (see for example [17–19]).

An efficient and swift analysis of an engineering system is called optimal analysis upon reaching sparse, well-conditioned and well-structured characteristics matrices of its mathematical model [20,21]. Optimal structural analysis is particularly effective for iterative analysis of large-scale structures. Recent advances and findings on graph theoretical matrix methods for optimal structural analysis in computational structural mechanics are comprehensively described in Refs. [20,22]. Numerous investigations performed on deterministic analysis of symmetric structures and cyclic symmetric structures [23], whereas it is rare on StFEM, based upon the best knowledge of the authors. Dynamics of cyclic symmetric structures was conducted by Thomas [24], and McDaniel and Chang [25]. Williams [26] presented an algorithm for exact eigensolution of rotationally periodic structures. Tran [27,28] utilized component mode method for vibration analysis of cyclic symmetry structures. He et al. [29,30] utilized scaled boundary finite element method for cyclically symmetric domain of heat transfer and structural mechanics problems. Group theoretic symmetry recognition for complex structural systems was investigated by Zingoni [31,32]. On the other hand, graph theory has provided useful technique for optimal analysis of structures [20–22]. Kaveh and Koohestani [33] formed graph models for regular finite element meshes. Kaveh and Rahami [34] applied block circulant matrices in free vibration analysis of cyclically repetitive structures. Koohestani and Kaveh [35] performed efficient buckling and free vibration analysis of cyclically repeated space truss structures. Kaveh and Rahami [36] proposed an efficient analysis of repetitive structures generated by graph products. Koohestani [37] presented an orthogonal self-stress matrix for efficient analysis of cyclically symmetric space truss structures using force method. Koohestani [38] exploited symmetry in graphs and applied to finite and boundary elements. The decomposition of generalized eigenproblems for the free vibration analysis of cyclically symmetric finite element models was implemented by Koohestani [39].

Stochastic finite element modeling is a time-consuming numerical tool requiring efficient analysis methods. This research proposes an efficient stochastic finite element analysis for regular structures. Two main strategies are presented for an optimal analysis in this paper. The first one employs a graph theoretical method for swift and symmetry-independent numerical solution of Fredholm integral equation arising from KLE, leading to substantial computational time reduction respect to ordinary numerical solutions. The second one develops a method based on recently proposed graph product rules [20,36,39] for stochastic finite element procedure of cyclic symmetric structures. A preconditioner is applied to decompose the matrices to Kronecker products of submatrices, and then graph product rules are formulated to solve the governing linear equation such that a small matrix is inversed rather than final large matrix (i.e., stiffness matrix). These mathematical remedies are incorporated to accelerate the analysis, while the accuracy is reserved. The proposed method provides significant improvement in the stochastic structural analysis. Numerical examples indicate the efficiency and accuracy of the present method as an optimal stochastic structural analysis.

2. Preliminaries on graph theory and applications

2.1. A few basic definitions from graph theory

Graph theory is a branch of discrete mathematics which has many engineering applications. Here, some necessary definitions and applications in structural mechanics are briefly explained according to the scope of this paper.

A graph S includes a non-empty set $N(S)$ of elements called *vertices* (nodes or points) and another set $M(S)$ of elements called *edges* (elements or arcs) together with a relation of incidence which associates with each member a pair of vertices (not necessarily distinct), called its ends. Two or more edges joining the same pair of vertices are known as multiple edges, and an edge joining a vertex to itself is called a loop. A graph with no loops and multiple edges is known as a *simple graph*. Two vertices of a graph are called adjacent if these vertices are the end vertices of an edge. The degree of a vertex of a graph is the number of edges incident with that vertex. A *complete graph* is a graph in which every two distinct vertices are connected by precisely one edge [20]. Different algebraic matrices exist for graph representation, among which the adjacency matrix $\mathbf{A}$ of a graph is needed to be defined here, as

$$a_{ij} = \begin{cases} 1; & \text{if node is } i \text{ adjacent to node } j \\ 0; & \text{otherwise} \end{cases} \tag{1}$$

Weighted graph is a graph whose vertices and edges are assigned by values called weights. Vertex weights and edge weights vectors are represented as

$$\begin{aligned} \mathbf{VW} &= [vw_i]; \quad i = 1,2,3,\ldots,N \\ \mathbf{EW} &= [ew_{ij}]; \quad (i,j) = 1,2,3,\ldots,N \end{aligned} \tag{2}$$

and the adjacency matrix of a weighted graph is

$$a_{ij} = \begin{cases} ew_{ij}; & \text{if node } i \text{ is adjacent to node } j \\ 0; & \text{otherwise} \end{cases} \tag{3}$$

which is attained by edge weights vector. In this study, the simple graphs are examined, and thus diagonal terms are zero as there are no loops.

2.2. Graph product rules for analysis of structures

In the graph theory, *graph products* is identified as binary operation on graphs focusing on regular and repetitive patterns and their properties. Therefore, a structure is called *regular* whenever the underlying model is a product of sub-graphs. The sub-graphs producing a product graph are known as its *generators*. Many structures with regular patterns may be viewed as the Cartesian product, strong Cartesian product or direct product of a number of simple graphs.

Graph product rules for analysis of repetitive structures has been presented in Ref. [36]. The first assumption belongs to the commutative property of two Hermitian matrices $\mathbf{A}_i$ and $\mathbf{A}_j$, which is necessary and sufficient condition for simultaneously diagonalization of these matrices using orthogonal matrix as follows

$$\mathbf{A}_i\mathbf{A}_j = \mathbf{A}_j\mathbf{A}_i; \quad i \neq j. \tag{4}$$

Now assume a matrix $\mathbf{Z}$ may be written as the sum of n Kronecker products as given by

$$\mathbf{Z} = \sum_{i=1}^{n} \mathbf{A}_i \otimes \mathbf{B}_i, \tag{5}$$

so that it can be transformed into a block diagonal matrix employing the mentioned condition. One can represent stiffness matrix of a cyclic symmetric structure in cylindrical coordinate system that