Computers and Structures 178 (2017) 71–87

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

An alternate stable midpoint quadrature to improve the element stiffness matrix of quadrilaterals for application of functionally graded materials (FGM)

Jeyakarthikeyan P.V. *, Subramanian G., Yogeshwaran R.

School of Mechanical Engineering, SRM University, Chennai 603203, India

ARTICLE INFO

Article history:
Received 28 July 2016
Accepted 4 October 2016

Keywords:
Quadrilateral element
Stiffness matrices
FGM
Richardson extrapolation
Hourglass instability
Patch test
Error estimation

ABSTRACT

An efficient, stable and accurate quadrilateral element and its improved stiffness matrix on the midpoint quadrature concept is proposed in this research study. As a first approximation, the integrating point is considered as midpoint of the element of the mapped 2-square in the (ξ, η) plane (same as one-point Gauss-Quadrature). As a second approximation or stabilizing function, integrating points are assumed to be either at the midpoint of the four quadrants or four element edges of the mapped 2-square element in the (ξ, η) plane and these interpolated data are assembled. An appropriate weighted addition of the two approximations is found to result in a better and stable stiffness matrix than equivalent time delayed value of Gauss Quadrature.

1. Introduction

When attempting to calculate the element stiffness matrices by FEM, a lot of effort is taken to determine as many simple representing sampling points as possible to diminish computational effort. Enough attention is to be given to develop efficient and simple numerical methods to solve the engineering problems in Finite element methods. At the same time this should not result in destroying the order of accuracy of FEM. Basically, it is noted that low order quadrature without stabilizing function, not only reduces the accuracy but also result in loss of stability unless it is handled with due diligence [1,2]. The various procedures are established including closed form solution as well as reducing the time required for numerical quadrature [3–8]. As long as geometric function (C° continues) to remain linear the closed form solution attempt is possible because Jacobian becomes matrix of constant. Subramanian et al. [9–12] discussed such universal matrix concept to evaluate closed form stiffness matrices for triangular and tetrahedron elements when Jacobian is a matrix of constant. The same method and principle extended to the family of both transition elements and infinite domain elements [13].

In present paper, two approximations of the stiffness matrices are calculated using the mid-point rule of two step sizes h and 0.5 h. It means the first approximation for stiffness matrix of the quadrilateral elements is sampled at the centroid {(0, 0) point} of the element and further in the second approximation for the step size of 0.5 h, either at the midpoints of the four quadrants ((−1/2, −1/2), (1/2, −1/2), (1/2, 1/2), and (−1/2, 1/2)) or midpoint of the four element boundary edges {(−1, 0), (0, −1), (0, 1), (1, 0)} of the mapped 2-square element in the (ξ, η) plane. Then the two approximations are extrapolated using Richardson extrapolation procedure to get a better result [14]. In the Proposed method, a similar mid-point rule is used to calculate the two approximations. As the error associated with it becomes proportional to h^2, the well-known Richardson extrapolation is used to get the aforementioned two approximations with different step size and this procedure gives the good convergence result [15,16]. Importantly, Richardson extrapolation [17] is most frequently used either to control the size of the error formed in the solution by the chosen numerical method or to attempt to increase the accuracy of the solution calculated by the original numerical methods.

Recent developments in modern improved material processing techniques have made it possible for manufacturers to provide a wide range of different better suited functionally graded materials (FGM). Such materials can be selected to have continuously varying material properties to suit specific requirements. Della Croce and Venini [18] discussed finite element formulation for such functionally graded plates. Kubair and Bhanu-Chandar [19] and Enab [20] addressed the problem of stress concentration factor due to

* Corresponding author.
E-mail address: jeyakarthik.p@ktr.srmuniv.ac.in (P.V. Jeyakarthikeyan).

http://dx.doi.org/10.1016/j.compstruc.2016.10.008
0045-7949/

a circular hole in functionally graded plate and they also discussed finite element formulation for this particular FGM. Kim and Paulino [21] discussed functionally graded finite elements for generalised isoparametric formulation and gave a comparison for treating the sub-element as homogeneous as well as graded element. Oliveira et al. [22] discussed the weighted quadrature rule for finite element method and proposed suitable weighting values for the alternate sampling points that were obtained by the general technique of solving polynomial approximation functions.

If the Gauss element is to be established for the FGM, usually, it samples are taken at the selected standard Gauss points for the material constant. So, in this stiffness matrix calculation more float point variables are involved and they increase the computational time proportionally. Therefore, the alternate proposed method in this study is intended to overcome these difficulties and offer simple, effective, hourglass controlled and time saving efficient methods to also handle FGM. Unlike the conventional Gauss Quadrature procedure, the proposed methods use only the integer values as typical sampling points. Hence, the calculation of stiffness matrix becomes very simple and effective without compromising the stability issues, while arriving at rate of convergence of the solution. These basic properties of the proposed elements are verified using some of the benchmark example problems and compared with Gauss Quadrature values.

2. Isoparametric element stiffness matrix formulation for non-homogenous materials (functionally graded materials-FGM)

$\prod$ is defined as the strain energy (U) minus the potential energy (W) of the loads

$$\prod = U - W \tag{1}$$

$$\mathrm{U} = \frac{1}{2}\int_V \underline{\sigma}^T \underline{\varepsilon}\, dV = \frac{1}{2}[u \quad v]^T [K][u \quad v] \tag{2}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-(1+\gamma)\mu)} \begin{bmatrix} 1-\gamma\mu & \mu & 0 \\ \mu & 1-\gamma\mu & 0 \\ 0 & 0 & \frac{1-(1+\gamma)\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \tag{3}$$

$$\{\sigma\} = [D]\{\varepsilon\} \tag{4}$$

where $(\sigma_{xx}\sigma_{yy}\tau_{xy})^{\mathrm{T}}$ and $(\varepsilon_{xx}\varepsilon_{yy}\varepsilon_{xy})^{\mathrm{T}}$ are the in-plane stresses and strains respectively. D is the material matrix which is a function of Young's Modulus (E) and Poisson's ratio (μ). The parameter γ describes either the plane stress (if,$\gamma = 0$) or plane strain (if,$\gamma = 1$) [19]. When the material properties are assumed to be constant, material matrix D is invariant with respect to the coordinate position of the elements. But in functionally graded materials, Young's Modulus and the Poisson's ratio are known as the function of coordinate position of the elements (x and y) in the domain as $E = E(x,y)$ and $\mu = \mu(x,y)$.

In the isoparametric formulation of quadrilateral elements (Fig. 1), the field variable function (u and v), geometric functions (x and y) and material functions (E and μ) are described as given below

$$u = \sum_{i=1}^{n} N(\xi,\eta)_i u_i \quad v = \sum_{i=1}^{n} N(\xi,\eta)_i v_i \tag{5}$$

$$x = \sum_{i=1}^{n} N(\xi,\eta)_i x_i \quad y = \sum_{i=1}^{n} N(\xi,\eta)_i y_i \tag{6}$$

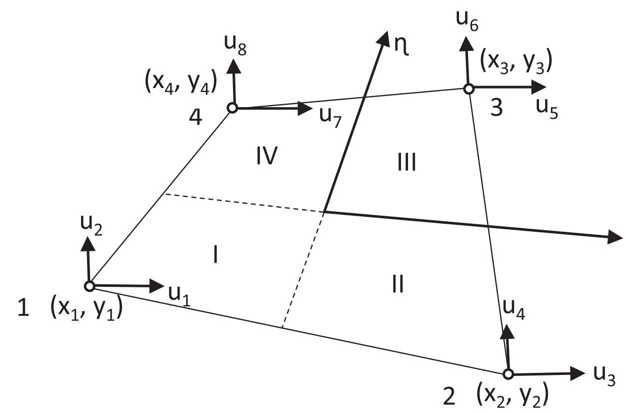

Fig. 1. Quadrilateral element: real and transform plane.

$$E = \sum_{i=1}^{n} N(\xi,\eta)_i E_i \quad \mu = \sum_{i=1}^{n} N(\xi,\eta)_i \mu_i \tag{7}$$

where n is number of nodes in the element and, N is the interpolation function of the elements.

The general Strain equation is,

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$$\{\varepsilon\} = [B] * [u \quad v]^T$$

the strain displacement matrix [B] can be written for plane stress and plain strain conditions as follows

$$B = (B_1)_{3\times4} * (B_2)_{4\times4} * (B_3)_{4\times2n}$$

$$[u \quad v]^T = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad \ldots \quad u_n \quad v_n]^T$$

where n represents number of nodes in the element

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} \frac{J_{22}}{|J|} & -\frac{J_{12}}{|J|} & 0 & 0 \\ -\frac{J_{21}}{|J|} & \frac{J_{11}}{|J|} & 0 & 0 \\ 0 & 0 & \frac{J_{22}}{|J|} & -\frac{J_{12}}{|J|} \\ 0 & 0 & -\frac{J_{21}}{|J|} & \frac{J_{11}}{|J|} \end{bmatrix}; \quad \text{and}$$

$$B_3 = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & . & . & \frac{\partial N_n}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & . & . & \frac{\partial N_n}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & . & . & 0 & \frac{\partial N_n}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & . & . & 0 & \frac{\partial N_n}{\partial \eta} \end{bmatrix}$$

Element stiffness matrix in the finite element method can be written as

$$[K^e] = \int_{-1}^{1}\int_{1}^{1} [B^e(\xi,\eta)]^T [D^e(\xi,\eta)][B^e(\xi,\eta)] \det[J^e] d\xi\, d\eta \tag{8}$$

To approximate the integration of stiffness matrix in two dimension as shown in Eq. (8), the appropriate numerical quadrature is used as follows:

$$[K^e] = \sum_{i=1}^{M}\sum_{j=1}^{N} W_i W_j K^e(\xi_i,\eta_j) d\xi\, d\eta$$

where M and N denote the number of quadrature points in the ξ and η directions and W_i and W_j represent the corresponding weights of the quadrature.