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Large deformation analysis of strain-gradient elastic beams

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ABSTRACT

A total Lagrangian finite element formulation for the analysis of large deformation of beams and frames, based on the strain-gradient elasticity theory and the Timoshenko beam model, is developed herein. A generalized version of the Kirchhoff-Saint Venant constitutive equation is proposed to capture geometric nonlinearities at small scales. Also, field variables are interpolated using C^1 shape functions for constructing conforming elements. Accordingly, a novel 6-DOF two-node beam element is introduced. To analyze frames, the formulation is extended so that a 9-DOF two-node frame element is produced. Several examples are studied to show the accuracy of proposed elements.

1. Introduction

The stiffness of the solids and structures predicted by the classical continuum mechanics is smaller than empirical results at small scales such that it may be said that the size of specimens plays a major role in the response of bodies [1]. In other words, at small scales, the displacement field predicted by the classical continuum theory has larger components than the experimental data for the same level of loading. Therefore, some gradient theories were developed to overcome the disadvantages of the classical one.

The strain-gradient elasticity theory is a special case of the theory of elasticity with microstructure developed by Mindlin [2]. Despite the classical elasticity, the potential energy density in this theory depends not only on the strain tensors, but also the gradient of kinematic tensors. By assuming infinitesimal deformations, three separate types for the theory have been developed by Mindlin [2] and Mindlin and Eshel [3]. It is worth noting that the strain-gradient theory of Mindlin [2] is the linearized version of the nonlinear couple-stress theory elaborated by Toupin [4]. In all forms of the strain-gradient theory which is defined in the linear isotropic regime, there are some non-classical material parameters in the potential energy density function. Generally, finding these unknowns is a challenging problem. Aifantis [5] proposed the simplest form of the strain-gradient theory, widely used in the literature [6–12], that includes only one non-classical constant, which is the so-called material length scale parameter.

Susmel et al. [13] compared the strain-gradient theory which is based on the model of Aifantis [5] with the theory of critical distances and derived a relation between the length scale parameter l in the strain-gradient theory and the critical distance value in theory of critical distances. Moreover, Bagni et al. [14] presented the material length scale parameter l in the strain-gradient elasticity for a quantity of materials.

In the field of moderately large deflections, Lazopoulos et al. [15] analyzed nonlinear bending and buckling of beams. Additionally, Lazopoulos and Lazopoulos [16] formulated the nonlinear elastic deformation of shallow shells. Ramezani [17] replaced the infinitesimal strains with the von-Karman strains in strain energy density function to take geometric nonlinearity of beams into account based on the Timoshenko model.

In the most above-mentioned studies, analytical methods have been used to solve differential equations of gradient elastic structures. Generally, analytical techniques often work on simple geometries and boundary conditions. Furthermore, due to appearance of the higher-order derivatives of field variables, dealing with the differential equations in the strain-gradient theory is much more complex than in classical mechanics. Consequently, a reliable numerical method is necessary to solve different problems. In addition, various branches of finite element method have been used in the past decades to solve linear problems in the gradient elasticity [18–24].

There are some practical applications in which beams experience large rotations and displacements, whereas strains are small. Hence, the analysis of finite deformation of beams is of great importance. In addition, a review of the literature shows that arbitrary large deformation analysis of beams and frames

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in strain-gradient media is still an open problem. Therefore, this research is devoted to geometrically nonlinear elastic deformation analysis of these structures under various geometries and loadings. Moreover, due to complexity of the governing equations, the finite element method is used to solve the highly nonlinear equilibrium static equations. Additionally, by producing strain-gradient-based finite elements for solids and structures, it is expected that the strain-gradient theory attracts practical applications.

The remainder of this research is arranged as follows. In the Section 2, some brief information on the strain-gradient theory at finite strains is presented. Also, kinematics of a Timoshenko beam is described and various quantities are calculated in the Section 3. Variational formulation of the problem is presented in the Section 4, where the weak form of the equations of equilibrium is derived. In the Section 5, the gradient-based finite element formulation is presented. A two-node beam element, with six degrees of freedom per node, is developed. Then, the beam element is generalized to beams with an arbitrary orientation in the plane leading to a frame element with nine degrees of freedom per node. In the Section 6, several numerical examples are analyzed to examine the accuracy and performance of the beam and frame elements introduced here. Finally, some conclusions are drawn from the present study in the Section 7.

2. The strain-gradient theory at finite deformation

In this section, a brief introduction to the generalization of the strain-gradient theory to finite, elastic deformations are presented. For more details on the subject, the pioneering works of Mindlin [2], Mindlin and Eshel [3], and Toupin [4] are suggested.

It is feasible to generalize the original, linear strain-gradient theory of Mindlin [2,3] to the large deformation. To this end, energy-density in the Form-II is rewritten by means of Green-Lagrange strain tensor $\mathbf{E}$ and its gradient $\boldsymbol{\Xi} = \nabla\mathbf{E}$ as follows [25–27]:

$$U(\mathbf{E},\boldsymbol{\Xi}) = \frac{1}{2}\lambda E_{ii}E_{jj} + \mu E_{ij}E_{ij} + a_1\Xi_{iik}\Xi_{kjj} + a_2\Xi_{ijj}\Xi_{ikk} + a_3\Xi_{iik}\Xi_{jjk} + a_4\Xi_{ijk}\Xi_{ijk} + a_5\Xi_{ijk}\Xi_{kji} \tag{1}$$

with

$$\mathbf{E} = \frac{1}{2}((\nabla\mathbf{u})^T + \nabla\mathbf{u} + \nabla\mathbf{u}(\nabla\mathbf{u})^T), \quad E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_m}{\partial X_i}\frac{\partial u_m}{\partial X_j}\right) \tag{2}$$

$$\boldsymbol{\Xi} = \nabla\mathbf{E}, \quad \Xi_{kij} = \frac{\partial E_{ij}}{\partial X_k} = \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial X_j \partial X_k} + \frac{\partial^2 u_j}{\partial X_i \partial X_k} + \frac{\partial^2 u_m}{\partial X_i \partial X_k}\frac{\partial u_m}{\partial X_j} + \frac{\partial u_m}{\partial X_i}\frac{\partial^2 u_m}{\partial X_j \partial X_k}\right) \tag{3}$$

where a_I $(I = 1-5)$ are five non-classical material parameters and λ and μ are the Lamé's constants. It is clear that the Lamé's constants may be expressed in terms of the Young's modulus E and the Poisson's ratio υ by the relations $\lambda = E\upsilon/[(1-2\upsilon)(1+\upsilon)]$ and $\mu = E/2(1+\upsilon)$. In addition, $\mathbf{u}$ is the displacement vector for the body and u_i $(i = 1–3)$ are its components in the rectangular coordinate system.

The constitutive equations by considering Eq. (1) are then given by

$$S_{ij} = S_{ji} = \frac{\partial U}{\partial E_{ij}} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij} \tag{4}$$

$$T_{ijk} = T_{ikj} = \frac{\partial U}{\partial \Xi_{ijk}} = \frac{1}{2}a_1(\delta_{ij}\Xi_{kpp} + 2\delta_{kj}\Xi_{ppi} + \delta_{ik}\Xi_{jpp}) + 2a_2\delta_{jk}\Xi_{ipp} + a_3(\delta_{ij}\Xi_{ppk} + \delta_{ik}\Xi_{ppj}) + 2a_4\Xi_{ijk} + a_5(\Xi_{jki} + \Xi_{kji}) \tag{5}$$

where δ_{ij} is the Kronecker delta and S_{ij}, and T_{ijk} are the components of the second Piola-Kirchhoff stress tensor $\mathbf{S}$ and the second Piola-Kirchhoff double-stress tensor $\mathbf{T}$, respectively. It is worthy of note that Eqs. (4) and (5) are the generalization of the classical Kirchhoff-Saint Venant constitutive equation.

3. Kinematics and kinetics of the strain-gradient Timoshenko beam model

In this section, kinematic assumptions on the motion of a particle deeming the Timoshenko beam model are first described. Afterwards, the components of the strain tensor $\mathbf{E}$ and the strain-gradient tensor $\boldsymbol{\Xi}$ are obtained, which can be used to calculate the components of the second Piola-Kirchhoff stresses, i.e. S_{ij} and T_{ijk}.

3.1. Beam geometry and kinematic assumptions

A prismatic beam which has uniform cross section along the X_1 direction is considered. Let L, h, and b be the length, height and width of the beam in the undeformed configuration, respectively. A rectangular Cartesian coordinate system $\{X_1, X_2, X_3\}$, with $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ as its orthonormal basis vectors, is deemed. The coordinate X_1 is along the length and X_2 is along the height of the beam, respectively. Moreover, X_3-axis is perpendicular to the X_1X_2-plane so that a right-handed orthogonal rectangular coordinate system is made. The beam cross section and external loading on the beam are further assumed to be symmetric with respect to the X_2-axis to avoid any torsional effects.

The position vector of a material point in the reference and current configurations are described by $\mathbf{X} = X_i\mathbf{E}_i$ and $\mathbf{x}$, respectively and the relation $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X},t)$ holds. In the Timoshenko beam model, it is assumed that plane cross sections, which are normal to the line of centroid in the undeformed configuration, remain plane, but not normal to the deformed, central line during deformation. Accordingly, the motion is described by

$$\boldsymbol{\varphi}(X_1, X_2, t) = \mathbf{x}_0(X_1, t) + X_2\mathbf{t}(X_1, t) \tag{6}$$

where $\mathbf{x}_0$ and $\mathbf{t}$ are the position vector of a material point on the deformed line of centroid and the director vector, respectively. These vectors are given by

$$\left.\begin{aligned} \mathbf{x}_0(X_1,t) &= [X_1 + u(X_1,t)]\mathbf{E}_1 + w(X_1,t)\mathbf{E}_2 \\ \mathbf{t}(X_1,t) &= -\sin\psi(X_1,t)\mathbf{E}_1 + \cos\psi(X_1,t)\mathbf{E}_2 \end{aligned}\right\} \tag{7}$$

In Eq. (7), the functions u and w denote the displacement components of a material point located on the line of centroid along the X_1 and X_2 directions, respectively. Additionally, the function ψ stands for the rotation angle of the beam cross section about the X_3-axis. Furthermore, the unit vector $\mathbf{n}$ perpendicular to the director vector as well as the deformed cross section of the beam is given by

$$\mathbf{n}(X_1,t) = \cos\psi(X_1,t)\mathbf{E}_1 + \sin\psi(X_1,t)\mathbf{E}_2 \tag{8}$$

Also, Fig. 1 includes the coordinate system and the kinematic quantities described above.

Notice that by substituting Eq. (7) into Eq. (6), the components of the displacement vector $\mathbf{u} = \mathbf{x} - \mathbf{X}$ may be expressed as