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A two-dimensional consistent approach for static and dynamic analyses of uniform beams

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ABSTRACT

A unified technique based on the scaled boundary finite element method is presented in this paper to analyze the bending, free vibration and forced vibration of thin to moderately thick beams with constant material properties and rectangular cross sections. The structure model is treated as a plane stress problem and the principle of virtual work involving the inertial force is applied to derive the scaled boundary finite element equation. Higher order spectral elements are used to discretize the longitudinal dimension and the solution through the thickness is expressed analytically as a Padé expansion. A variable transformation technique facilitates the development of the dynamic stiffness matrix, which leads to the static stiffness and mass matrices naturally. Rayleigh damping and Newmark-β method are employed to perform the forced vibration analysis. Numerical examples covering static and dynamic analyses validate the excellent performance and capability of this approach.

1. Introduction

Beam member, as one of the most important structural components, is widely used in various engineering applications. A considerable amount of effort has been devoted to investigating its bending and vibration behaviors over the past years.

The classical Euler–Bernoulli theory has long been recognized as a convenient approximation for slender beams and serves as a cornerstone for structural analysis and design [1]. However, the model becomes invalid for non-slender beams and tends to slightly overestimate the natural frequencies especially for higher modes. Based on previous developments, an improved beam theory was proposed by Timoshenko [2,3] which takes into account the rotary inertia and shear deformation. The subsequent emergence of the finite element method brought about overwhelming interest in the numerical implementation of the Timoshenko theory due to its relatively wide applicability and C° continuity requirement. Earlier Timoshenko beam finite elements [4–8] differ from each other either in the choice of displacement field variables and the corresponding interpolation functions or in the formulation techniques used to develop the finite element models. Particularly, when linear shape functions are used for both transverse displacement and rotation, the element behaves very stiff in the thin beam limit. Such phenomenon is known as shear locking whose source was shown to be the mathematical operations involved in the shape function definitions and subsequent integration of functionals [9]. Obviously, shear locking restricted the application of Timoshenko beam finite elements to some extent. Special and continuous attention has henceforth been focused to overcome the locking problem or to develop other more efficient beam elements.

Reduced integration and selective reduced integration [9–11] are apparently the earliest attempts to remedy the deficiency of the shear deformable beam elements. Unfortunately, such techniques fail to ensure complete removal of spurious constraints [9]. From the perspective of dynamic analysis, reduced integration improves the convergence for the bending dominated spectra but introduces additional errors in the shear energy terms which in turn reduces the convergence rate of the frequencies of the shear dominated spectra [12]. In comparison, field consistent method [9,13,14] is a more fundamental treatment of locking phenomenon because it enables the elements to reproduce only the true constrained strain states. Nevertheless, these elements cannot lead to the two-node superconvergent element [15–17] though shear locking is successfully avoided. The superconvergent beam element is based on interdependent interpolations of transverse and rotational displacements and possesses the superconvergent character for static problems. When it comes to dynamic cases, however, the locking-free element exhibits slow convergence in predicting flexural frequencies and would not represent pure shear frequencies accurately [17].

Mixed formulations provide alternative approaches to circumvent the shear locking suffered by conventional Timoshenko beam finite elements. It has been shown that certain classes of mixed formulations are actually equivalent to the displacement models with reduced integration scheme [11]. An assumed strain-displacement beam model was

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developed by Reddy [17] based on a variational form where the displacements and strains are treated as independent field variables. More generally, from a three-field variational form based on an extension of Hu–Washizu principle Taylor et al. [18] proposed a finite element method for the solutions to Timoshenko beams. Contrary to the more traditional displacement-related methods mentioned above, a force-based formulation was recently employed to combine a higher order beam theory to construct an innovative beam finite element [19]. The model is inherently free from shear locking and exhibits other promising capabilities.

The finite element formulations based on higher order beam theories [20,21], in which the requirement for a shear coefficient is eliminated, are preferable for avoiding shear locking. Heyliger and Reddy [22] developed a higher order beam finite element and studied the bending and vibrations of isotropic beams. Subsequently, a unified finite element model [23] that contains the Euler-Bernoulli, Timoshenko and simplified Reddy third-order beam theories as special cases was presented. The beam element does not experience locking and the general stiffness matrix can be specialized to any of the three theories by merely assigning proper values to the parameters involved in the development. Progressively, in the framework of Carrera's Unified Formulation (CUF) refined beam elements [24] were proposed for more general analyses, where CUF offers a systematic procedure to obtain refined structural models by considering the order of the theory as a free parameter. Naturally, both classical beam elements and higher order beam elements can be derived as particular cases of the hierarchical formulation. These CUF models were afterwards extended to the dynamic response analysis of slender structures [25].

It is well known that locking can be alleviated by increasing the polynomial degree of the approximations. The correlated work for beam situations was initiated by Li [26] who analyzed the *p* and *hp* versions of the finite element method and proved error estimates independent of the aspect ratio of the beam. Spectral elements and hierarchical elements, which both belong to the *p*-type finite elements but differ in the adoption of element shape functions, were recently examined and employed to perform the dynamic analysis of Timoshenko beams [27,28]. In addition, a family of discontinuous Galerkin methods originated by Celiker et al. [29] was devised for the Timoshenko beam problem. These methods were then shown to be free from locking through the error analysis for their *hp* versions [30]. In contrast with the aforementioned Galerkin approximation procedures, isogeometric collocation methods [31] opened another door to dealing with shear locking. Originally, da Veiga et al. [32] developed locking-free mixed formulations for straight planar Timoshenko beams in the context of isogeometric collocation approaches. After that single-variable isogeometric collocation and Galerkin methods for the Timoshenko beam problem were established by Kiendl et al. [33]. These formulations are completely locking-free and involve less degrees of freedom compared with standard Timoshenko beam elements.

Above is a rough literature review on various numerical techniques for locking-free beam analyses in statics and dynamics. In addition to applicability, however, other important aspects including accuracy, convergence, computational efficiency and the complexity level of mathematical formulation should also be taken into account when we evaluate a certain numerical approach. Unfortunately, none of these methods is found to be distinctly superior to the rest in all aspects. The fact stimulates ongoing interest in developing more general, more efficient and more robust numerical methods for analyzing beam structures.

The scaled boundary finite element method (SBFEM) [34] is a semi-analytical procedure which combines the advantages of the finite element method and the boundary element method and also exhibits appealing features of its own. This has been demonstrated by its successful applications to various engineering problems associated with soil-structure interaction [35–38], unbounded domains [39], stress singularities [40,41], crack propagation [42,43], potential flow [44], wave-structure interaction [45], liquid sloshing [46,47], structural dynamics [48], acoustics [49,50], viscoelasticity [51], elastoplasticity [52,53], thin-walled beams [54], static electricity [55], wave propagation [56], heat conduction [57], stress analysis [58,59] and stochastic analysis [60]. Particularly, the SBFEM was employed to establish a unified technique [61] for plate bending analysis which is based on the three-dimensional linear elastic theory and does not exhibit shear locking. The technique was subsequently improved [62] by substituting the original Taylor expansion for Padé expansion in approximating the relevant matrix exponential to construct the stiffness matrix more efficiently. This modified procedure has been applied to analyze piezoelectric plates [63] and magneto-electro-elastic plates [64,65]. However, all the aforementioned studies focus only on the static behaviors of plate structures. The first attempt for dynamic analysis of structures using the numerical approach in [62] was made by Xiang et al. [66] who investigated the free vibration and mechanical buckling of composite plates. Unfortunately, the mass matrix and the geometric stiffness matrix were derived by the conventional finite element method, different from the manner in which the static stiffness matrix was constructed. Additionally, it is obvious that the bending of beams has not been examined in the framework shown in [62], let alone their dynamic behaviors.

The objective of this paper is to present a truly consistent technique for the dynamic analyses of thin and moderately thick beams by extending the technique shown in [62]. The structure model is handled as a plane stress problem and the corresponding scaled boundary finite element equation is derived by applying the principle of virtual work that incorporates the work contributed by the inertial force, which develops the formulation concept in [62]. Higher order spectral elements are used to discretize the longitudinal dimension of the beam and the solution through the thickness is expressed analytically as a Padé expansion. A variable transformation technique facilitates the development of the dynamic stiffness matrix, which produces the static stiffness and mass matrices naturally. The numerical example on free vibration analysis demonstrates that the present method is superior to the one shown in [66] in predicting natural frequencies of structures. Both static and dynamic examples on regular beams with various slenderness ratios are investigated to validate the accuracy and applicability of the proposed approach. The forced vibrations of two stepped beams are also explored to highlight the capability and computational efficiency of the present technique.

The remainder of this paper is structured as follows. Section 2 presents the formulation of the basic problem. The corresponding solution procedure is provided in Section 3. Following is Section 4 which contains both static and dynamic numerical examples. Concluding remarks are given in Section 5.

2. Problem formulation

2.1. Basic representation

A regular beam with length L, constant height h and width b is shown in Fig. 1. The z and x coordinates of the Cartesian coordinate system are chosen along the transverse and longitudinal directions of the beam, respectively. Note that the beam is treated as a plane stress problem. Let the time-varying generalized displacements $\{u(x, z, t)\}$ at any point within the structure be expressed as the column vector

$$\{u\} = \left[u_z\ u_x\right]^{\mathrm{T}} \tag{1}$$

where u_z and u_x are the displacement components along the z- and x-directions. The strain vector $\{\varepsilon\}$ is expressed as [67]

$$\{\varepsilon\} = \left[\varepsilon_z\ \varepsilon_x\ \gamma_{xz}\right]^{\mathrm{T}} = [L]\{u\} \tag{2}$$

with the differential operator

$$[L] = \begin{bmatrix} \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \end{bmatrix} \tag{3}$$