Engineering Analysis with Boundary Elements 82 (2017) 57–67

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Finite block method in fracture analysis with functionally graded materials

J. Li [a], J.Z. Liu [b,*], T. Korakianitis [c], P.H. Wen [d,*]

[a] School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, Hunan, China
[b] School of Mathematics and Physics, Hunan Institute of Technology, Hengyang, Hunan, China
[c] Parks College of Engineering, Aviation and Technology, Saint Louis University, St. Louis, MO 63103, USA
[d] School of Engineering and Materials Science, Queen Mary, University of London, London E1 4NS, UK

ARTICLE INFO

Keywords:
Finite block method
Mapping technique and differential matrix
Lagrange series interpolation
Mixed-mode stress intensity factors
Functionally graded material
Anisotropy

ABSTRACT

The finite block method (FBM) is developed to determine stress intensity factors with orthotropic functionally graded materials under static and dynamic loads in this paper. By employing the Lagrange series, the first order partial differential matrix for one block is derived with arbitrary distribution of nodes. The higher order derivative matrix for two dimensional problems can be constructed directly. For linear elastic fracture mechanics, the COD and *J*-integral techniques to determine the stress intensity factors are formulated. For the dynamic problems, the Laplace transform method and Durbin's inverse technique are employed. Several examples are given and comparisons have been made with both the finite element method and analytical solutions in order to demonstrate the accuracy and convergence of the finite block method.

1. Introduction

In 1980s, the concept of functionally graded material was firstly proposed. Because of the variation in composition and structure gradually over volume, it results in corresponding changes in the properties of the material. The first characteristic of FGM is that the variation of material properties in FGM can be pre-determined by controlling the spatial distribution of the composition and the volume fraction of their constituents. Therefore, these materials benefit from the performance of its constituents, such as the high temperature and corrosion resistance of ceramics on one side, and large mechanical strength and toughness of metals on the other side [1-2]. For the realistic problems, such as the transient heat conduction in anisotropic and non-homogeneous media, it becomes a complex task and the mathematical modelling due to their complexity for both analytical and numerical analysis.

There are two categories in numerical engineering, i.e. the domain-type and boundary-type methods. Domain-type methods including the finite element method (FEM) and the boundary-type methods include boundary element method (BEM) can be found in the references (see Zienkiewicz et al. [3], Aliabadi [4] and Atluri [5]). Although the BEM is one of the most accurate and efficient methods, the fundamental solutions or Green's functions are required. For 2D and 3D dynamic problems in homogeneous and anisotropic solids, the fundamental solutions are very rarer in closed forms. Even in the static cases, there are very few applications to problems in anisotropic materials. In addition, the governing equations for FGM composites contain many coordinates and directions dependent coefficients, see Sladek et al [6].

For exponentially graded non-homogeneous, isotropic and linear elastic solids, the fundamental solutions have been derived by Marin and Lesnic [7] for three dimensions statics and Chan et al [8] for two dimensions statics with the analytical solutions expressed in some complicated finite integrals. The Fourier-integral representations of the elastodynamic fundamental solutions have been recently derived by Zhang et al. [9,10] for fracture analysis in FGMs. Due to the mathematical complexities for the non-homogeneous nature of FGMs, only a few investigations on the transient dynamic responses of cracked FGMs can be found in journals including the dynamic responses under impact loading investigated by Babaei and Lukasiwicz [11], Li and Zou [12]. In addition, FGMs exhibits isotropic or anisotropic material properties which depend on the processing technique and the engineering requirements. In recent years, meshless formulations are more and more popular due to their high adaptive and low cost to prepare input and output data in numerical analysis [13,14,15]. Sladek et al [16] extended the meshless method of the local Petrov-Galerkin approach to the stress analysis in two-dimensional anisotropic and linear elastic/viscoelastic solids with continuously varying material properties. Jin and Paulino [17] investigated a crack in a viscoelastic strip of FGM under tensile load. The stress intensity factors with mixed-mode are obtained in viscoelastic FGMs with correspondence principle. Kim and Paulino [18] presented a general purpose FEM formulation and implementation for linear FGMs and

* Corresponding authors.
E-mail addresses: hnhyls@126.com, p.h.wen@qmul.ac.uk (P.H. Wen).

http://dx.doi.org/10.1016/j.enganabound.2017.05.012
Received 9 March 2017; Received in revised form 14 May 2017; Accepted 22 May 2017
0955-7997/

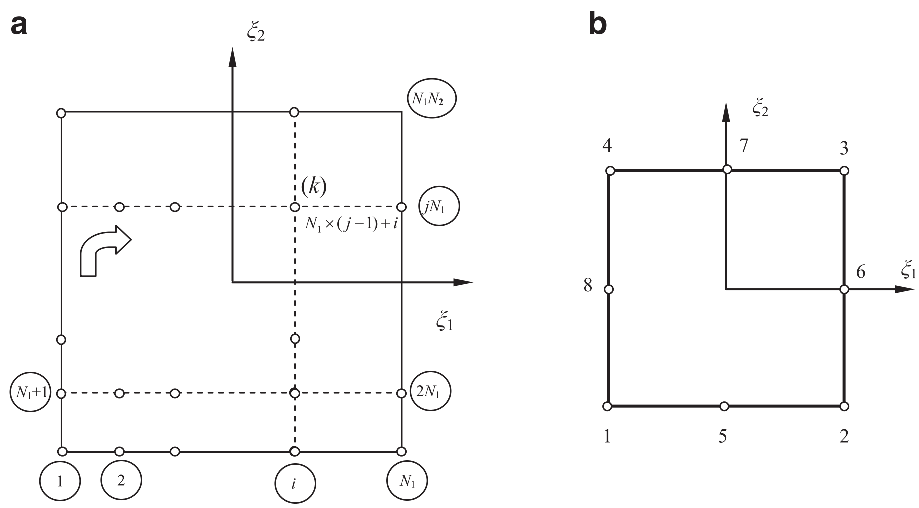

Fig. 1. Two-dimensional node distribution in mapping domain: (a) the local number system of node; (b) square domain with eight seeds for the mapping geometry.

fracture of FGMs for mixed-mode cracks by using COD and *J*-integral techniques.

The finite block method based on the point collocation method was developed firstly to solve the heat conduction problem in the functionally graded media and anisotropic materials by Li and Wen [19]. This method has been applied to nonlinear elasticity including contact and fracture mechanics successfully by Wen et al [20] and Li et al [21,22]. The essential feature of the FBM is that the physical domain is divided into few blocks only and the partial differential matrices are applied for each block which is similar to FEM, i.e. the domain is divided into several blocks with continuity conditions of stress and displacement on the interfaces. It is easy to prove that all stress components are continuous along the interface along the interfaces between two blocks. With the first order derivative matrices, the higher orders of partial differential matrices can be obtained in a straight forward manner. The quadratic type of block is transformed to normalised domain with eight seeds for 2D and then the partial differential matrices in physical domain are obtained using the differential matrices in the normalised domain. A set of algebraic equation from equilibrium equations in strong form is formulated in term of the nodal values of displacement. For linear elastic fracture mechanics, the static and dynamic stress intensity factors are evaluated by crack opening displacement (COD) and *J*-integral technique for both isotropic and orthotropic FGMs. To demonstrate the accuracy and efficiency of the FBM, several numerical examples are given with comparisons made with the finite element method and the local Petrov–Galerkin approach.

2. Two dimension differential matrices

Consider a set of nodes shown in Fig. 1 (normalised domain) with the nodes collocated at $\xi_\alpha^{(k)}$, $\alpha = 1, 2$, $k = 1, 2, ..., N_\alpha$, where N_α are numbers of nodes along two axes. By two dimension Lagrange interpolation polynomials, the function $u(\xi_1, \xi_2)$ can be approximated by

$$u(\xi_1,\xi_2) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} F_1(\xi_1,\xi_1^{(i)})F_2(\xi_2,\xi_2^{(j)})u^{(k)} \tag{1}$$

where

$$F_\alpha(\xi_\alpha,\xi_\alpha^{(i)}) = \prod_{\substack{m=1\\ m\neq i}}^{N_\alpha} \frac{(\xi_\alpha-\xi_\alpha^{(m)})}{(\xi_\alpha^{(i)}-\xi_\alpha^{(m)})} \tag{2}$$

and the superscript in Eq. (1) $k = (j-1)\times N_1 + i$, the number of nodes in total is $M = N_1 \times N_2$. Then the first order partial differential is determined easily with respects to ξ_1

$$\frac{\partial u}{\partial \xi_1} = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \frac{\partial F_1(\xi_1,\xi_1^{(i)})}{\partial \xi_1} F_2(\xi_2,\xi_2^{(j)})u^{(k)} \tag{3}$$

where

$$\begin{aligned}\frac{\partial F(\xi_1,\xi_1^{(i)})}{\partial \xi_1} &= \frac{\partial}{\partial \xi_1}\prod_{\substack{m=1,\\ m\neq i}}^{N_1}\frac{(\xi_1-\xi_1^{(m)})}{(\xi_1^{(i)}-\xi_1^{(m)})}\\ &= \sum_{l=1}^{N_1}\prod_{k=1,k\neq i,k\neq l}^{N_1}(\xi_1-\xi_1^{(k)}) \Big/ \prod_{m=1,m\neq i}^{N_1}(\xi_1^{(i)}-\xi_1^{(m)})\end{aligned} \tag{4}$$

and

$$\frac{\partial u}{\partial \xi_2} = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \frac{\partial F_2(\xi_2,\xi_2^{(j)})}{\partial \xi_2} F_1(\xi_1,\xi_1^{(i)})u^{(k)} \tag{5}$$

where

$$\begin{aligned}\frac{\partial F_2(\xi_2,\xi_2^{(j)})}{\partial \xi_2} &= \frac{\partial}{\partial \xi_2}\prod_{\substack{m=1,\\ m\neq i}}^{N_2}\frac{(\xi_2-\xi_2^{(m)})}{(\xi_2^{(j)}-\xi_2^{(m)})}\\ &= \sum_{l=1}^{N_2}\prod_{k=1,k\neq i,k\neq l}^{N_2}(\xi_2-\xi_2^{(k)}) \Big/ \prod_{m=1,m\neq j}^{N_2}(\xi_2^{(j)}-\xi_2^{(m)})\end{aligned} \tag{6}$$

For a block in the real domain, the mapping technique is introduced. In general, for two-dimensional area Ω in the Cartesian coordinate (x_1, x_2) can be mapped into a square Ω' in the domain $(\xi_1, \xi_2)|\xi_1| \le 1$; $|\xi_2| \le 1$ by using a set of quadratic shape functions with eight seeds. The quadratic shape functions are defined below

$$N_i(\xi_1,\xi_2) = \frac{1}{4}(1+\xi_1^{(i)}\xi_1)(1+\xi_2^{(i)}\xi_2)(\xi_1^{(i)}\xi_1+\xi_2^{(i)}\xi_2-1) \text{ for } i = 1,2,3,4 \tag{7}$$

$$N_i(\xi_1,\xi_2) = \frac{1}{2}(1-\xi_1^2)(1+\xi_2^{(i)}\xi_2) \text{ for } i = 5,7 \tag{8}$$

$$N_i(\xi_1,\xi_2) = \frac{1}{2}(1-\xi_2^2)(1+\xi_1^{(i)}\xi_1) \text{ for } i = 6,8 \tag{8}$$

The coordinate transform can be written as

$$x_\alpha = \sum_{k=1}^{8} N_k(\xi_1,\xi_2)x_\alpha^{(k)} \tag{9}$$