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Numerical analysis of an advective diffusion domain coupled with a diffusive heat source

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ABSTRACT

This paper presents a formulation of the boundary element method (BEM) for the study of heat diffusion and advective effect in isotropic and homogeneous media. The proposed formulation has a time independent fundamental solution obtained from the two-dimensional Laplace equation. Consequently, the formulation is called D-BEM since it has domain integrals in the basic integral equation. The first order time derivative that appears in the integral equations is approximated by a backward finite difference scheme. Homogeneous subregions are considered in the analysis in a specific model simulating a nonuniform flow passing by a circular obstacle under internal heat generation. Results from numerical models are compared with the available analytical solutions. The correlation estimator R^2 is employed to validate the numerical model and to demonstrate the accuracy of the proposed formulation.

1. Introduction

In engineering it is common to apply numerical methods for solving complex real problems governed by differential equations. By using appropriate models, parametric analyses allow understanding the problem under several different conditions by simply changing the model parameters. It is a cost effective approach if compared to experimental modeling.

The boundary element method (BEM) is known to be a powerful numerical technique formulated from integral equations for solving various computational mechanics problems [1,2]. It has been used to address an increasing number of problems in solid mechanics [2], fluid dynamics and acoustics [3,4], electromagnetic imaging [5], cathodic protection [6], elastodynamics [7], among others. Also, in many cases, coupled to other numerical methods [8–10]. A thorough historical review was presented by Cheng and Cheng [11].

Concerning the topic of this paper, current literature shows numerous works involving the BEM for the solution of advection and diffusion problems. Among them, Jesus and Azevedo [8], Jesus and Pereira [12] and Vanzuit [9]. In [8,9] are present solutions for the dynamic problem of heat diffusion, adopting an independent fundamental solution time marching schemes in time based on finite differences, beyond using the Houbolt method in this first work and the Hammer method in the second work with the use of cells to approximate the domain integrals, and also. In [12] is present a two-dimensional flow analysis in porous media using continuous homogeneous subregions in a stationary case based on the Laplace equation.

In [13] a BEM formulation is presented using time independent fundamental solution for the advection–diffusion steady state analysis. For the transient analyses the authors used a time dependent fundamental solution. DeSilva et al. [14] proposed solutions for the two dimensional advection–diffusion problems with variable velocities, using a time dependent fundamental solution and cell domain integrations.

Also with the use of cell integration, Lima Jr. et al. [15] numerically analyzed the mechanical behavior of continuous and saturated porous media with an implicit BEM formulation, with a time independent fundamental solution. In this work, the authors suggested a fluid structure interaction formulation, adopting, a numerical Gauss integration procedure on the boundary elements and a semi-analytical scheme for the domain cells. Other transient heat diffusion problems have also been also analyzed by others [16–18], with time dependent fundamental solutions.

Loeffler and Costalonga [19] used dual reciprocity to solve diffusive–advective problems, varying the velocity of flow and analyzing the impact on energy transport on the thermal diffusion. Also using reciprocity, Ochiai [20] presented a two-dimensional analysis of unsteady heat diffusion using a time independent BEM formulation and heat generation. In this work the author shows that it is possible to obtain satisfactory temperature distributions with the use of low order fundamental solutions. Guo et al. [21] presented a formulation to solve three-dimensional

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conduction problems with transient heat generation. In their work, the time dependence on the problem was temporarily removed from the equations by Laplace transform, preserving the boundary integral equation and avoiding the domain discretization. The use of reciprocity was also observed in the work of Tanaka et al. [22], which showed a BEM approach for two-dimensional conduction problems of transient heat transfer in anisotropic media. Their work made use of a time independent fundamental solution for isotropic materials and time marching scheme based on finite differences.

A formulation of an alternative boundary element method based on an exponential transformation variable to stabilize diffusion–advection problems is presented in [23], converting the equation of diffusion–advection into a modified Helmholtz equation. In this paper the authors discuss three transformations and differentiate its use for problems dominated by diffusion and advection.

Analyses of transient heat conduction problems without domain discretization were presented by Sutradhar and Paulino [24], transforming an inhomogeneous problem in a homogeneous diffusion problem with the Laplace transform and Galerkin approximations. In this work the time dependence is restored by numerical inversion of Laplace transformation through Stehfest algorithm [25]. The results obtained with the adopted formulation were compared with the solutions obtained from finite element simulations. In [26] the method of separation of variables and the principle of Duhamel is used to transform the one-dimensional problem of diffusion and heat generation in a reverse analysis problem based on the BEM.

Transient heat conduction problems using radial integration in the BEM formulation was analyzed by Yu et al. [27].

In this analysis the authors solved the problem of heat conduction with variable thermal conductivities.

In all cited studies, the BEM is used to obtain an approximate solution of the problem and the coupling with other methods (e.g. finite difference) is a common practice. In this work, the numerical study is focused on the problem of heat diffusion and advection. Two dimensional BEM formulations with time independent fundamental solutions are coupled to solve a problem of a circular diffusive heat generating source within an advective diffusion domain. Numerical results are compared to analytical ones to validate the developed codes.

2. Mathematical model

The advective–diffusion equation in a two dimensional isotropic and homogeneous domain Ω with boundary Γ is written as [28]

$$\frac{\partial u(X,t)}{\partial t} = -\nabla \bullet [\mathbf{v}(X)\,u(X,t)] + \frac{1}{Pe}\nabla^2 u(X,t) \qquad X \in \Omega,\ X = (x,y) \tag{1}$$

where Pe is the Peclét number, defined as

$$Pe = |\,\mathbf{v}\,|\frac{B}{\alpha} \tag{2}$$

in which $\mathbf{v}$ is the velocity vector (m/s), B is the characteristic length (m), α represents the coefficient of thermal diffusivity measured in m^2/s, u is the temperature, X is the field point coordinate and t is the time variable.

The essential and natural boundary conditions are, respectively:

$$u(X,t) = \hat{u}(X,t) \quad X \in \Gamma_u \tag{3}$$

$$q(X,t) = \frac{\partial u(X,t)}{\partial n(X)} = \hat{q}(X,t) \quad X \in \Gamma_q \tag{4}$$

and the initial condition at $t = t_0$ is given by

$$u(X,t) = u_0(X,t_0) \quad X \in \Omega \tag{5}$$

3. D-BEM formulation

3.1. Advection–diffusion equation

The integral equation of the D-BEM formulation for the advective–diffusion equation can be written as follows [28]

$$\begin{aligned} C(\xi)u(\xi,t) = {} & \int_\Gamma u*(\xi,X)\,q(X,t)\,d\Gamma - \int_\Gamma q*(\xi,X)\,u(X,t)\,d\Gamma \\ & -Pe\int_\Omega \frac{\partial u(X,t)}{\partial t}\,u*(\xi,X)\,d\Omega \\ & -Pe\int_\Omega \nabla\bullet[\mathbf{v}(X)\,u(X,t)]\,u*(\xi,X)\,d\Omega \\ & X\in\Omega,\ X=(x,y) \end{aligned} \tag{6}$$

where $C(\xi)$ is a geometric coefficient at the collocation point ξ, q is the thermal flux and u^* and q^* are the fundamental solution and its normal derivative, respectively. The term $\mathbf{v}(X)$ represents the time independent velocity field.

The fundamental solution $u^*(\xi, X)$ in the D-BEM formulation is independent of the time variable and is given by [29],

$$u*(\xi,X) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \tag{7}$$

where $r = |X - \xi|$ is the distance between field and collocation points.

The derivative of the fundamental solution with respect to the normal direction to the boundary is given by

$$q*(\xi,X) = \frac{\partial u*}{\partial r}\frac{dr}{dn} = -\frac{1}{2\pi\, r}\frac{dr}{dn} \tag{8}$$

where n is the outward direction normal to the boundary.

The kinetic term, $\nabla\bullet[\mathbf{v}(X)\,u(X,\ t)]$, in the domain integral of Eq. (6) can be written as

$$\nabla\bullet[\mathbf{v}(X)\,u(X,t)] = \mathbf{v}(X)\bullet\nabla u(X,t) + u(X,t)\nabla\bullet\mathbf{v}(X) \tag{9}$$

The first term in the right side of Eq. (9) represents the thermal gradient due to transport fluid mass with velocity $\mathbf{v}$ and the second term is the temperature established by the velocity gradient. Thus, the integral containing $\nabla\bullet[\mathbf{v}(X)\,u(X,\ t)]$ takes the following form

$$\begin{aligned} \int_\Omega \nabla\bullet[\mathbf{v}(X)\,u(X,t)]\,u*(\xi,X)\,d\Omega = {} & \int_\Omega \mathbf{v}(X)\bullet\nabla u(X,t)\,u*(\xi,X)\,d\Omega \\ & + \int_\Omega u(X,t)\,\nabla\bullet\mathbf{v}(X)\,u*(\xi,X)\,d\Omega \end{aligned} \tag{10}$$

For simplicity, the time derivative presented in Eq. (6) is approximated by the backward finite difference formula [30]

$$\frac{\partial u(X,t)}{\partial t} = \frac{u(X,t+\Delta t) - u(X,t)}{\Delta t} \tag{11}$$

Replacing Eqs. (10) and (11) in Eq. (6) and grouping terms conveniently, one has

$$\begin{aligned} C(\xi)u(\xi,t+\Delta t) = {} & \int_\Gamma u*(\xi,X)\,q(X,t+\Delta t)\,d\Gamma - \int_\Gamma q*(\xi,X)\,u(X,t+\Delta t)\,d\Gamma \\ & -Pe\left[\int_\Omega \mathbf{v}(X)\bullet\nabla u(X,t+\Delta t)\,u*(\xi,X)\,d\Omega \right. \\ & \left. + \int_\Omega u(X,t+\Delta t)\,\nabla\bullet\mathbf{v}(X)\,u*(\xi,X)\,d\Omega\right] \\ & -\frac{Pe}{\Delta t}\left[\int_\Omega u(X,t+\Delta t)\,u*(\xi,X)\,d\Omega - \int_\Omega u(X,t)\,u*(\xi,X)\,d\Omega\right] \\ & X\in\Omega,\ X=(x,y) \end{aligned} \tag{12}$$

Eq. (12) can be used recursively for the solution of advective–diffusion problems, starting with known variables at time t_τ and determining the unknown variables at time $t_{\tau+1}$. According to [31], the time step, Δt_c, can be estimated as

$$\Delta t_c \le \frac{{L_j}^2}{2\,\alpha} \tag{13}$$