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Galerkin boundary integral formulation for axisymmetric stokes flow

A. Xue [a], E. Graciani [b], L.J. Gray [c,*], V. Mantič [b], Maria Garzon [d]

[a] Oak Ridge High School, Oak Ridge, TN 37830 Present address: Duke University, Durham NC 27708, USA
[b] Elasticity and Strength of Materials Group, University of Seville, Camino de los Descubrimientos, Seville 41092, Spain
[c] 119 Berwick Drive, Oak Ridge, TN 37830, USA
[d] Department of Applied Mathematics, University of Oviedo, Oviedo, Spain

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ABSTRACT

A Galerkin boundary integral formulation for 3D axisymmetric Stokes flow is presented. The singular integrals are evaluated by splitting the complicated Green's function kernels into a singular term that can be integrated analytically, plus a term for which Gauss quadrature provides sufficient accuracy. As in a previous axisymmetric Laplace implementation, the treatment of the additional on-axis singularity is aided by employing a modified Galerkin weight function, and a similar splitting method is then employed to handle this singularity. The target application of the Stokes algorithm is to model the breakup of one viscous fluid enclosed inside a second, and this two fluid problem can be formulated in terms of a single boundary integral equation along the interface. The Galerkin form for this equation is derived herein.

1. Introduction

This paper presents a Galerkin boundary integral formulation for axisymmetric Stokes flow [1]. The motivation for this work is to study, via numerical simulations, the breakup of viscous fluids [2]. Although the breakup of inviscid fluids, *i.e.*, the Rayleigh–Taylor instability for potential flow [3,4], dominates the literature, viscous flow is significant for a variety of industrial applications [5]. In particular, viscous flow is important in small regions relevant to capillary flow, hemodialysis, blood virus removal, and tertiary oil recovery [6,7]. In addition, droplet formation in viscous fluids is involved in ink-jet printing techniques used for the production of integrated circuits, specialized optical lenses, and microarrays for DNA analysis [5].

Of the two main numerical methods for solving boundary integral equations, namely collocation and Galerkin, the latter approximation is more involved and computationally slower. It is therefore not surprising that, to our knowledge, all previously reported axisymmetric Stokes algorithms have employed the simpler collocation scheme [1,8–11]. The motivation for developing a Galerkin formulation is the past success with this technique in the modeling inviscid breakup, and a goal of this work is to carry out the corresponding viscous simulations. For potential flow, a Galerkin boundary integral analysis coupled with a Level Set method [12] was successful in tracking fluid motion up to and beyond pinch-off in both single fluid [13–15] and two-fluid systems [16]. Most notably, for the two-fluid system, characterized by the ratio of the outer and inner fluid densities $D = \rho_{ext}/\rho_{int}$, the Galerkin approach obtained scaling exponents over the entire range $0 < D < \infty$ [16], and the exponents were in good agreement with the experimental results of Burton and Taborek [17]. However, collocation boundary integral approximations (coupled with marker particle front tracking) failed for $D > 6$ [18]: the fluid interface exhibited unphysical 'roll-up instabilities' (see Fig. 1), and there was no scaling. While it is not clear whether the failure was due to collocation, the front tracking algorithm, or both, Section 5 will derive the Galerkin form for the interface equation that governs the two-fluid system.

Regarding this two fluid system, it is also worth noting that a boundary integral analysis has several important advantages compared to a volume formulation. First, it is clearly simpler to remesh just the fluid interface as the fluid domains evolve. Second, the exterior fluid equation is posed in an infinite domain, and as demonstrated in Section 5, this much easier to handle with boundary integrals.

Key to the boundary integral formulation is the axisymmetric Stokes (velocity) Green's function and the corresponding traction kernel. These fundamental solutions have appeared in many forms, and indeed the aforementioned collocation papers [1,8–11,19] all have different formulas. Herein the Green's function expressions will follow that of Graziani [9,10] and Graciani [20,21]. With one major notational change discussed below, these two references are basically in agreement. It should also be noted that [20,21] is concerned with axisymmetric elasticity; however, the Stokes flow equations are obtained by simply setting the

* Corresponding author.
E-mail address: harpogray@gmail.com (L.J. Gray).

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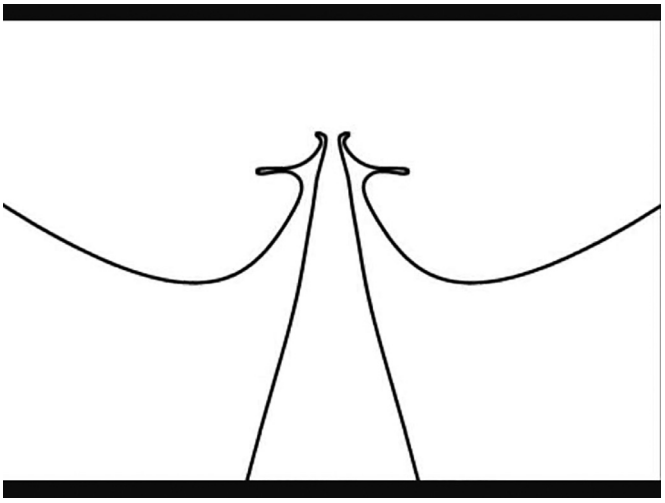

Fig. 1. Time snapshot close to pinch-off from a collocation/marker particle simulation of two fluid potential flow. For $D > 6$, D the ratio of the inner and outer fluid densities, the computed fluid interface displays an unphysical 'roll-up instability'.

Poisson ratio to $\nu = 0.5$, the shear modulus equal to the viscosity, and identifying the displacements as the fluid velocities.

As with the Laplace equation, the axisymmetric Stokes Green's functions involve complete elliptic integrals of the first and second kind and therefore include logarithmic singular terms. In the Laplace implementation [22], the Green's function expressions were simple enough that the entire logarithm singularity

$$\int_0^1 f(t)\log(t)\mathrm{d}t \tag{1}$$

could be handled with a special Gauss quadrature technique. While this approach would be possible for the Stokes flow problem, the kernel functions are sufficiently complicated that this formulation would be quite cumbersome, and, moreover, would likely lead to a very inefficient calculation. As discussed in Section 3, the approach adopted herein is to split the singular integrals: the most singular terms, namely $\log(t)$ and $t\log(t)$ (at $t = 0$) are integrated analytically, while remaining powers $t^k\log(t)$, $k \geq 2$ can be integrated numerically with sufficient accuracy. The somewhat extensive algebra involved in achieving this decomposition can be handled with symbolic computation, *e.g.*, Maple.

2. Galerkin boundary integral equation

The conversion of the three-dimensional Stokes equations for an incompressible viscous fluid into an axisymmetric boundary integral equation can be found in many places, *e.g.*, [1,8,9]. With $\boldsymbol{u}(Q)$ and $\boldsymbol{\tau}(Q)$ the surface velocity and traction, this integral equation can be written in the form

$$\mathcal{P}(\hat{r},\hat{z}) \equiv \int_\Gamma r\Big(\boldsymbol{\mathcal{T}}(Q,P)\boldsymbol{u}(Q) - \boldsymbol{\mathcal{U}}(Q,P)\boldsymbol{\tau}(Q)\Big)\mathrm{d}\Gamma_Q = 0, \tag{2}$$

and formulas for the Green's function kernels $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{T}}$ are given in Section 2.1. Here $Q = (r, z)$, $P = (\hat{r}, \hat{z})$ are boundary points, and it is assumed that the $\boldsymbol{\mathcal{T}}$ singular integral is defined as a limit from outside the domain [22]. This boundary limit process replaces the commonly used Cauchy Principal Value, and automatically incorporates the 'free term' that commonly appears in the boundary integral statement. Finally, employing a cylindrical coordinate system $\{r, \theta, z\}$, with z the symmetry axis, the problem domain reduces to a 2D contour in the $\{r, z\}$ plane, see Fig. 2.

Following [22], the Galerkin formulation of Eq. (2) is

$$\int_\Gamma \hat{r}\hat{\psi}_k(\hat{r},\hat{z})\mathcal{P}(\hat{r},\hat{z})\mathrm{d}\Gamma_P = 0. \tag{3}$$

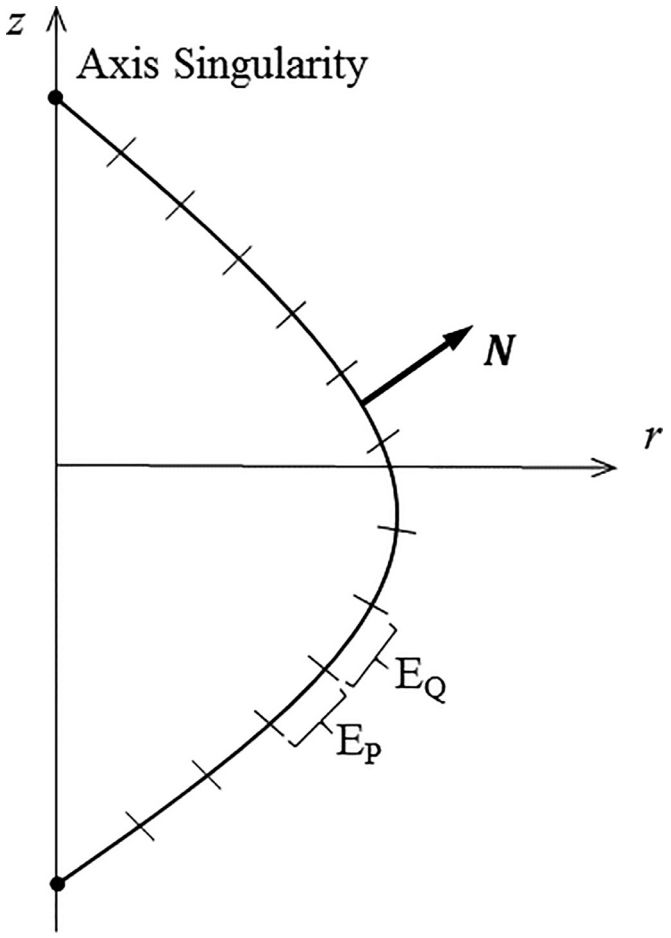

Fig. 2. Illustration of a 2D geometry for a 3D axisymmetric boundary integral calculation. The two marked elements depict the adjacent singular case for a Galerkin approximation, while the axis points $r = 0$ are highlighted to note the axis singularity. The normal vector shown is the exterior normal for the interior fluid.

Here the standard Galerkin weight function $\hat{\psi}_k(\hat{r},\hat{z})$ is comprised of the (usually two) element shape functions (defined below) that are nonzero at the boundary node P_k. The extra factor of $\hat{r}$ restores symmetry, as $\hat{r}\boldsymbol{\mathcal{U}}$ is symmetric (and a Symmetric Galerkin formulation would be possible), but the key advantage of including this factor is that $\hat{r} = 0$ on the symmetry axis. There is an additional singularity when $Q = P$ is on the axis, and this causes some difficulty for collocation approximations [20,21]. As will be discussed in Section 3.3.3, with Galerkin and the modified weight function, this singularity can be handled in a straightforward manner.

Herein Eq. (3) will be approximated using standard linear interpolation of the boundary and boundary functions. The two-dimensional linear shape functions are

$$\begin{aligned}\psi_1(t) &= 1 - t\\ \psi_2(t) &= t.\end{aligned} \tag{4}$$

$t \in [0, 1]$, while the geometry and function interpolations on an element with nodes P_1 and P_2 are

$$\begin{aligned}P(t) &= \psi_1(t)P_1 + \psi_2(t)P_2\\ \boldsymbol{u}(P(t)) &= \psi_1(t)\boldsymbol{u}(P_1) + \psi_2(t)\boldsymbol{u}(P_2)\\ \boldsymbol{\tau}(P(t)) &= \psi_1(t)\boldsymbol{\tau}(P_1) + \psi_2(t)\boldsymbol{\tau}(P_2).\end{aligned} \tag{5}$$

2.1. Green's functions

As noted in the Introduction, a variety of formulas and notations have been employed for the Green's function kernels. We mainly follow