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A boundary element method for a class of elliptic boundary value problems of functionally graded media

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ABSTRACT

A Boundary Element Method (BEM) is derived for obtaining solutions to a class of elliptic boundary value problems (BVPs) of functionally graded media (FGM). Some particular examples are considered to illustrate the application of the BEM.

1. Introduction

Whereas the BEM provides an effective numerical procedure for the solution of BVPs for homogeneous media the same is not generally true for inhomogeneous media. In the case of the inhomogeneous media, the material is assumed to be a functionally graded material, i.e., the material properties vary spatially according to known smooth functions. BVPs for such media have governing equations with variable coefficients. A BEM for 2D diffusion-convection problems in homogeneous anisotropic media has been recently considered by Haddade, Salam, Khaeruddin and Azis in [17]. In recent years some progress toward finding numerical solutions to BVPs for FGM by using BEM has been made. Clements [2], Cheng [4,5], Rangogni [6], Shaw [7], Gipson et al. [8], Ang et al. [9], and Clements and Azis [10] considered the case for isotropic FGM.

In the case of anisotropic FGM there are few published studies. BVPs which are relevant for certain classes of problems for anisotropic FGM have been considered by Azis and Clements [11], Azis et al. [12], Azis and Clements [13], Azis and Clements [14,15]. An elliptic equation which is also relevant for a certain class of problems for anisotropic FGM has been considered by Clements and Rogers [1]. They obtained a boundary integral equation for the case when the coefficients in the equation depend on one spatial variable only. Specifically the equation considered by Clements and Rogers [1] takes the form

$$\frac{\partial}{\partial x_i}\left[\lambda_{ij}(x_2)\,\frac{\partial\phi(x_1,x_2)}{\partial x_j}\right]=0$$

where the coefficients λ_{ij} depend on x_2 only and the repeated summation convention (summing from 1 to 2) is employed.

This paper is concerned with obtaining boundary integral equations for the solution of BVPs governed by equations of the form

$$\frac{\partial}{\partial x_i}\left[\lambda_{ij}(x_1,x_2)\,\frac{\partial\phi(x_1,x_2)}{\partial x_j}\right]=0 \tag{1}$$

Equations of this type govern the behavior of a wide class of BVPs of both isotropic and anisotropic FGM. Antiplane strain in elastostatics and plane thermostatics for anisotropic FGM are two areas for which the governing equation is of the type (1).

Several techniques will be considered for obtaining boundary integral equations for the solution of (1). For each technique it is necessary to place some constraint on the class of coefficients λ_{ij} for which the solution obtained is valid. Some numerical examples are considered to illustrate the application of the boundary integral equations. The analysis of this paper is purely formal; the main aim being to construct effective BEMs for classes of equations which fall within the type (1).

2. The boundary value problem

Referred to a Cartesian frame Ox_1x_2 a solution to (1) is sought which is valid in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\phi(\mathbf{x})\big(\mathbf{x}=(x_1,x_2)\big)$ is specified and on $\partial\Omega_2$

$$P(\mathbf{x})=\lambda_{ij}\left(\partial\phi/\partial x_j\right)n_i \tag{2}$$

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is specified where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to $\partial\Omega$.

For all points in Ω the matrix of coefficients $[\lambda_{ij}]$ is a real symmetric positive definite matrix so that throughout Ω Eq. (1) is a second order elliptic partial differential equation. Further, the coefficients λ_{ij} are required to be twice differentiable functions of the two independent variables x_1 and x_2.

The method of solution will be to obtain boundary integral equations from which numerical values of the dependent variables ϕ and P may be obtained for all points in Ω. The analysis here is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = 0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

3. Reduction to a constant coefficient equation

The coefficients λ_{ij} are required to take the form

$$\lambda_{ij}(\mathbf{x}) = \lambda_{ij}^{(0)}\, g(\mathbf{x}) \tag{3}$$

where the $\lambda_{ij}^{(0)}$ are constants and g is a differentiable function of x. Use of (3) in (1) yields

$$\lambda_{ij}^{(0)} \frac{\partial}{\partial x_i}\left(g \frac{\partial \phi}{\partial x_j} \right) = 0 \tag{4}$$

Let

$$\psi(\mathbf{x}) = g^{1/2}(\mathbf{x})\, \phi(\mathbf{x}) \tag{5}$$

so that (4) may be written in the form

$$\lambda_{ij}^{(0)} \frac{\partial}{\partial x_i}\left[g \frac{\partial \left(g^{-1/2}\psi\right)}{\partial x_j} \right] = 0$$

That is

$$\lambda_{ij}^{(0)}\left[\left(\frac{1}{4} g^{-3/2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{1}{2} g^{-1/2}\frac{\partial^2 g}{\partial x_i \partial x_j}\right)\psi + g^{1/2}\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right] = 0 \tag{6}$$

Use of the identity

$$\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = -\frac{1}{4} g^{-3/2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j} + \frac{1}{2} g^{-1/2}\frac{\partial^2 g}{\partial x_i \partial x_j}$$

permits (6) to be written in the form

$$g^{1/2}\lambda_{ij}^{(0)}\frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi\, \lambda_{ij}^{(0)}\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = 0$$

It follows that if g is such that

$$\lambda_{ij}^{(0)}\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + k g^{1/2} = 0 \tag{7}$$

then the transformation (5) carries the variable coefficients Eq. (4) to the constant coefficients equation

$$\lambda_{ij}^{(0)}\frac{\partial^2 \psi}{\partial x_i \partial x_j} + k\psi = 0 \tag{8}$$

where k is a constant.

Also, substitution of (3) and (5) into (2) gives

$$P = -P^{[g]}\psi + P^{[\psi]} g^{1/2} \tag{9}$$

where

$$P^{[g]}(\mathbf{x}) = \lambda_{ij}^{(0)}\frac{\partial g^{1/2}}{\partial x_j} n_i \qquad P(\mathbf{x}) = \lambda_{ij}^{(0)}\frac{\partial \psi}{\partial x_j} n_i$$

A boundary integral equation for the solution of (8) is given in Clements [3] in the form

$$\eta(\mathbf{x}_0)\, \psi(\mathbf{x}_0) = \int_{\partial\Omega} \left[\Gamma(\mathbf{x}, \mathbf{x}_0)\, \psi(\mathbf{x}) - \Phi(\mathbf{x}, \mathbf{x}_0)\, P^{[\psi]}(\mathbf{x})\right] ds(\mathbf{x}) \tag{10}$$

where $\mathbf{x}_0 = (a, b)$, $\eta = 0$ if $(a,\ b) \notin \Omega \cup \partial\Omega$, $\eta = 1$ if $(a,\ b) \in \Omega$, $\eta = \frac{1}{2}$ if $(a,\ b) \in \partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at $(a,\ b)$.

The so called fundamental solution Φ in (10) is any solution of the equation

$$\lambda_{ij}^{(0)}\frac{\partial^2 \Phi}{\partial x_i \partial x_j} + k\Phi = \delta(\mathbf{x} - \mathbf{x}_0)$$

and the Γ is given by

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \lambda_{ij}^{(0)}\frac{\partial \Phi(\mathbf{x}, \mathbf{x}_0)}{\partial x_j} n_i$$

where δ is the Dirac delta function. For two-dimensional problems Φ and Γ are given by

$$\Phi(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi}\ln R & \text{if } k = 0 \\ \frac{\imath K}{4} H_0^{(2)}(\omega R) & \text{if } k > 0 \\ \frac{-K}{2\pi} K_0(\omega R) & \text{if } k < 0 \end{cases}$$

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi}\frac{1}{R}\lambda_{ij}^{(0)}\frac{\partial R}{\partial x_j} n_i & \text{if } k = 0 \\ \frac{-\imath K\omega}{4} H_1^{(2)}(\omega R)\lambda_{ij}^{(0)}\frac{\partial R}{\partial x_j} n_i & \text{if } k > 0 \\ \frac{K\omega}{2\pi} K_1(\omega R)\lambda_{ij}^{(0)}\frac{\partial R}{\partial x_j} n_i & \text{if } k < 0 \end{cases} \tag{11}$$

where

$$\begin{aligned} K &= \ddot{\tau}/\zeta \\ \omega &= \sqrt{|k|/\zeta} \\ \zeta &= [\lambda_{11}^{(0)} + \lambda_{12}^{(0)}(\tau + \overline{\tau}) + \lambda_{22}^{(0)}\tau\overline{\tau}]/2 \\ R &= \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2} \\ \dot{x}_1 &= x_1 + \dot{\tau} x_2 \\ \dot{a} &= a + \dot{\tau} b \\ \dot{x}_2 &= \ddot{\tau} x_2 \\ \dot{b} &= \ddot{\tau} b \end{aligned}$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\lambda_{11}^{(0)} + 2\lambda_{12}^{(0)}\tau + \lambda_{22}^{(0)}\tau^2 = 0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. K_0, K_1 denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one and the bar denotes the complex conjugate. A technique for finding the fundamental solution Φ in Eq. (11) may be found in Azis [16].

The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (11) are given by

$$\begin{aligned} \frac{\partial R}{\partial x_1} &= \frac{1}{R}(\dot{x}_1 - \dot{a}) \\ \frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}(\dot{x}_1 - \dot{a})\right] + \ddot{\tau}\left[\frac{1}{R}(\dot{x}_2 - \dot{b})\right] \end{aligned}$$

Use of (5) and (9) in (10) yields

$$\begin{aligned} \eta(\mathbf{x}_0)\, g^{1/2}(\mathbf{x}_0)\, \phi(\mathbf{x}_0) = \int_{\partial\Omega} \Big\{ &\left[g^{1/2}(\mathbf{x})\, \Gamma(\mathbf{x}, \mathbf{x}_0) - P^{[g]}(\mathbf{x})\, \Phi(\mathbf{x}, \mathbf{x}_0)\right]\phi(\mathbf{x}) \\ &- \left[g^{-1/2}(\mathbf{x})\, \Phi(\mathbf{x}, \mathbf{x}_0)\right] P(\mathbf{x}) \Big\} ds(\mathbf{x}) \end{aligned}$$

This equation provides a boundary integral equation for determining ϕ and P at all points of Ω.

The analysis of the section requires that the coefficients λ_{ij} are of the form (3) with g satisfying (7). This condition on g allows for considerable choice in the coefficients. For example, when $k = 0$, g can assume a number of multiparameter forms with the parameters being employed