Engineering Analysis with Boundary Elements 84 (2017) 191–205

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Radial basis functions methods for boundary value problems: Performance comparison

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ARTICLE INFO

Keywords:
Radial basis functions
Compactly supported
Collocation method
Galerkin method
Subdomain collocation

ABSTRACT

We present in this paper comparisons on the performances among five typical radial basis functions methods, namely radial basis collocation method (RBCM), radial basis Galerkin method (RBGM), compactly supported radial basis collocation method (CSRBCM), compactly supported radial basis Galerkin method (CSRBGM), and finite subdomain radial basis collocation method (FSRBCM), for solving problems arising from engineering industries and applied sciences. Numerical comparison results demonstrate that the RBCM and FSRBCM possess high accuracy and superior convergence rates in which the FSRBCM particularly attains higher accuracy for problems with large gradients. The FSRBCM, CSRBCM and RBCM are computationally efficient while the CSRBCM, CSRBGM and FSRBCM can greatly improve the ill-conditioning of the resultant matrix. In conclusion, its advantages on high accuracy; exponential convergence; well-conditioning; and effective computation make the FSRBCM a first-choice among the five radial basis functions methods.

1. Introduction

In the last decades, the development of meshfree methods (meshless methods) establishes the prominence in computational mechanics due to their distinct advantages on eliminating the need of element connectivity for input data required in most conventional mesh based methods and greatly cutting down the modeling cost as mesh generation is time-consuming and labor intensive. Overviews of general meshfree methods and their performances in engineering applications can be found in [1–11]. Generally, there are two typical patterns for the Meshfree formulations: weak form Galerkin method [1]–[5] and strong form collocation method [8]–[11]. For the approximation, a broad variety of functions can be employed as the interpolation function. In addition to polynomial, spline functions etc., particularly, we pay attention to radial basis functions (RBFs).

A geophysicist Hardy [12],[13] was the first to introduce Multiquadric (MQ) RBFs for the scattered data interpolation in the early 1970s, which were proved to be the best interpolation functions in more than thirty functions [14]. Investigation of the existence and uniqueness of MQ interpolants provided the theoretical justification for the success of MQs [15],[16]. Further, MQ interpolation error was reported to converge at exponential rate [16],[17]. Polyharmonic splines [18] are also powerful functions for data interpolation that have a general algebraic convergence [19],[20], in which a special class in even dimension is called thin-plate spline (TPS) [21],[22]. Duchon [22] explored the existence and uniqueness of the TPS interpolation and introduced a family of semi-norms to show its convergence. Another popular RBF is Gaussian which was first proposed by a geophysicist Krige [23] for the data interpolation and further developed by Matheron [24]. Wendland [25] refined the error bound for Gaussian which was known for spectral convergence as MQs. An overview of the local error estimates for the commonly used RBFs interpolation of scattered date can refer to Wu and Schaback [26]. Other RBFs without popularity include Sobolev splines [27], Markoff function [23] etc. Some summary work of using RBFs for the data interpolation can be found in [28],[29].

The great success of introducing RBFs to solve partial differential equations (PDEs) brings them in many engineering applications. This concept was initiated by Kansa [9,30] using MQs associated with collocation method to solve parabolic, hyperbolic and elliptic equations, and its theoretical foundation was presented by Franke and Schaback [31]. Following Kansa's method, Cecil et al. [32] constructed a numerical scheme for Hamilton–Jacobi equations in arbitrary dimensions. Utilizing random collocation points with RBFs approximation was developed by Zerroukat et al. [33] for the heat transfer problem. RBFs with collocation also perform well in solving the time dependent problems [34–37], singularity problems [38],[39], composite material problems [40],[41], inverse problems [42], and many other applications [43–48]. A summary can refer to Fornberg and Flyer [49]. Cheng et al. [50] established an exponential error estimate of the collocation solution adopting the global MQs and Gaussian for the PDEs. Wendland [51] combined

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http://dx.doi.org/10.1016/j.enganabound.2017.08.019
Received 7 May 2017; Received in revised form 26 August 2017; Accepted 29 August 2017
0955-7997/

Galerkin method with global and local RBFs, and derived error estimates of this method to solve PDEs, which led to the same error bounds in the energy norm as the classical finite element method (FEM). Another popular RBF method formulated in Galerkin weak form is radial point interpolation method [52],[53] whose shape function has delta function property.

Conventional RBFs are generally globally supported and the resulting matrices are poorly conditioned. One usually tries to remedy these conundrums by fine tuning of the shape parameter, domain partition or preconditioning, and localization etc. Decreasing shape parameter improves the ill-conditioning quite easily, but also brings down the accuracy. Volokh and Vilney [54] came up with a truncated singular value decomposition (TSVD) to transform a very ill-conditioned RBF asymmetric collocation system into a well-conditioned reduced system, which worked effectively as a preconditioner [55],[56]. Condition number can also be greatly reduced by the domain decomposition [58],[59]. Ling and Hon [57] formulated an affine space decomposition scheme which was particularly pleasant as it is also stable for very large complex systems of PDEs. Moreover, Chen, Wang and their coworkers [39,60],[61] proposed a subdomain radial basis collocation method which could not only well improve the ill-condition of the resultant matrix by constructing a sparse matrix but also prevent the iterations between subdomains which was unavoidable in domain decomposition methods.

Another well-used technique for alleviating the ill-conditioning of the interpolation matrix is to localize the global RBFs. Schaback and Wendland [62] introduced the first instance of such functions and further developments were provided by Wu [63],[64] who created numerous piecewise polynomial compactly supported radial basis functions (CSRBFs). Wendland [65] constructed more positive definite and compactly supported radial functions which consisted of a univariate polynomial within their support. These radial functions are impressive by their simple forms. A localized RBF enhanced by reproducing kernel was proposed by Chen et al. [66] within collocation method which enjoyed the exponential convergence but introduced more degrees of freedom. Other localized RBFs can be found in [67]–[69]. Wendland [70] investigated the error estimate of the interpolation by CSRBFs, which showed that the convergence rate of the interplant to the prescribed target function was related to the smoothness of the approximating localized RBFs.

This article is to contrast five representative RBFs methods: RBFs associated with collocation method, called radial basis collocation method (RBCM) [[9],[10]]; RBFs associated with Galerkin method, called radial basis Galerkin method (RBGM) [51]; CSRBFs in conjunction with collocation method, termed compactly supported radial basis collocation method (CSRBCM) [31]; CSRBFs in conjunction with Galerkin method, termed compactly supported radial basis Galerkin method (CSRBGM) [77]; and finite subdomain radial basis collocation method (FSRBCM) employing RBFs with sundomain collocation [61]. Besides, RBFs are also prevalent approximation functions within many other methods, such as boundary knot method [71], MLPG method [72] and finite difference method [73] etc. The paper is organized as follows. In Section 2, we present an overview of global RBFs and CSRBFs. In Section 3, the RBFs interpolation and their error estimates are described. Five archetypal RBFs methods for interpolation and solving PDEs and the corresponding error estimates are investigated in Section 4. Numerical solutions of these five methods are compared in Section 5 and the concluding remarks are followed in Section 6.

2. Radial basis functions

2.1. Global radial basis functions

Multivariate functions which can be expressed as univariate functions of the Euclidean norm $\|\cdot\|$ are call radial functions. RBFs are a special class of functions which can be efficiently evaluated and they are positive definite or conditionally positive definite. Their characteristic is that the response only depends on the distance from a central point and they are globally nonlocal functions.

A typical RBF is MQ RBF proposed by Hardy [12],[13] expressed as

$$\phi(r) = \left(r^2 + c^2\right)^{n-\frac{3}{2}}, \qquad n = 1, 2, 3, \dots \tag{1}$$

where $r = \|\mathbf{x} - \mathbf{x}_I\| (I = 1, 2, \dots, N_s)$ denotes the radial distance from the center, $\mathbf{x}_I = (x_{1I} \quad x_{2I} \quad x_{3I})$ is the source point of RBF and N_s is the number of source point. c is the shape parameter which controls the intensity of the functions, and it has profound influence on the solution accuracy and convergence of the RBFs approximation. Larger c yields flatter shape function which is insensitive to the difference in the radial distance and leads to higher accuracy. However, at the same time, the resulting matrix becomes more ill-conditioned. Therefore, an optimum c exists for a solution by balancing the approximation accuracy and ill-condition of the matrix. When $n = 1$, MQ monotonically decreases with distance from the center and increases for otherwise n. The function is called reciprocal MQ (or inverse MQ) when $n = 1$, linear MQ when $n = 2$, cubic MQ when $n = 3$, and so forth. MQs are of special interest because they are superior interpolation functions that cannot be matched by other functions [14] and have spectral convergence property [16] which will be described hereinafter.

Gaussian RBF [23] decreases with distance from the center monotonically as inverse MQ, which reads

$$\phi(r) = \exp\left(-\frac{r^2}{c^2}\right) \tag{2}$$

Gaussian RBF is popular because it is more local in which the response is only vital in a neighborhood near the center than MQs that have a global response for such kind of problems with locality property. Nevertheless, similar behavior of c and convergence property of Gaussian RBF can be detected as that of inverse MQ.

Polyharmonic splines are another class of representative RBFs described as follows

$$\phi(r) = \begin{cases} r^{2k-d} \log r, & \text{if } d \text{ is even} \\ r^{2k-d}, & \text{if } d \text{ is odd,} \end{cases} \quad k > \frac{d}{2} \tag{3}$$

where $k \in \mathbb{N}$ is any natural number and d is the dimension of the space. Especially, when $\phi(r) = r$ the function is called multiconic, when $\phi(r) = r^3$ it's called pseudocubic or cubic [22], and when $\phi(r) = r^n \log r$, it's also entitled logarithmic. As the degree k is increased, the shape function becomes flatter and better accuracy can be attained. Particularly, when $2k - d = 2$, this special polyharmonic spline in even space is called Thin plate splines [21],[22]

$$\phi(r) = r^2 \log r \tag{4}$$

TPS is another RBF in common use because it's a great interpolation function next to MQs [14].

2.2. Compactly supported radial basis functions

Since the RBFs mentioned above are global, the corresponding matrix is a full matrix, and the evaluation demands the whole set of summands. To achieve a sparse matrix, therefore, a family of CSRBFs is introduced. It was initially constructed by Wu [63] and later expanded by Wendland [65] and Buhmann [67],[68] in the mid 1990s. Generally, a CSRBF takes the form

$$\phi(r) = (1 - r)_+^p p(r) \tag{5}$$

in which

$$(1 - r)_+^p = \begin{cases} (1 - r)^p, & if\ r \le 1 \\ 0 \quad , & otherwise \end{cases} \tag{6}$$

where $p(r)$ is a prescribed polynomial, $r = \|\mathbf{x} - \mathbf{x}_I\| / \vartheta \, (I = 1, 2, \dots, N_s)$ is the Euclidean distance and ϑ is the dilatation parameter which controls the size of the support. Wu [64] first proposed a class of CSRBFs as