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An efficient numerical technique for solution of two-dimensional cubic nonlinear Schrödinger equation with error analysis

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ABSTRACT

In this paper, the spectral meshless radial point interpolation (SMRPI) technique is applied to the solution of two-dimensional cubic nonlinear Schrödinger equations. Firstly, we obtain a time discrete scheme by approximating time derivative via a finite difference formula, then we use the SMRPI approach to approximate the spatial derivatives. This method is based on a combination of meshless methods and spectral collocation techniques. The point interpolation method with the help of radial basis functions is used to construct shape functions which act as basis functions in frame of SMRPI. In the current work, the thin plate splines (TPS) are used as the basis functions and in order to eliminate the nonlinearity, a simple predictor-corrector (P-C) scheme is performed. We prove that the time discrete scheme is unconditionally stable and convergent in time variable using the energy method. We show that convergence order of the time discrete scheme is $\mathcal{O}(\delta t)$. The aim of this paper is to show that the SMRPI method is suitable for the treatment of the nonlinear Schrödinger equations. Also, the SMRPI has less computational complexity than the other methods that have already solved this problem. The results of numerical experiments are compared with analytical solution to confirm the accuracy and efficiency of the presented scheme.

1. Introduction

The present paper considers the following generalized cubic nonlinear two-dimensional Schrödinger equation with wave function $u(\mathbf{x}, t)$:

$$i\frac{\partial u(\mathbf{x},t)}{\partial t} + \alpha\Delta u(\mathbf{x},t) + \beta|u(\mathbf{x},t)|^2 u(\mathbf{x},t) + w(\mathbf{x})u(\mathbf{x},t) = 0,$$
$$\mathbf{x} \in \Omega, t \in (0, T], \tag{1}$$

with initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2}$$

and Dirichlet boundary condition

$$u(\mathbf{x}, t) = h(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, t > 0, \tag{3}$$

where $u(\mathbf{x}, t)$ is an unknown complex function, $i = \sqrt{-1}$, $\Omega \subset \mathbb{R}^2$, $\partial\Omega$ is the boundary of Ω, $u_0(\mathbf{x})$ and $h(\mathbf{x}, t)$ are given sufficiently smooth functions and $w(\mathbf{x})$ is an arbitrary potential function that is real value and bounded on Ω. The cubic nonlinear Schrödinger equation occurs in a variety of areas, including, quantum mechanics [1], nonlinear optics [2,3], electromagnetic wave propagation [4].

In optics, the nonlinear Schrödinger equation occurs in the Manakov system, a model of wave propagation in fiber optics. The function u represents a wave and the nonlinear Schrödinger equation describes the propagation of the wave through a nonlinear medium. The second-order derivative represents the dispersion, while the β term represents the nonlinearity. The equation models many nonlinearity effects in a fiber, including but not limited to self-phase modulation, four-wave mixing, second harmonic generation, stimulated Raman scattering, etc [5–9].

Since the solutions of the Schrödinger equation is very important for describing several phenomena in physics and engineering, therefore solving this equation is necessary. As a short list that numerically investigated the solutions of various aspects of the Schrödinger equation, we refer to some of them. In [10–13], they have applied various types of the finite difference schemes, in [14–20] they have employed some meshfree methods and in [21–23] they have used the alternating direction implicit (ADI) schemes for solving several models of linear and nonlinear Schrödinger equations. As references that have investigated analytical solutions, can be cited to [24,25]. In [24], the author has used an effective method namely the improved tan $(\Phi(\xi)/2)$-expansion method for constructing a range of exact solutions for the following partial differential equation

$$iu_t + u_{xx} \pm 2\gamma|u|^2 u = 0, \tag{4}$$

where γ is a non-zero real constants and $u = u(x, t)$ is a complex-valued function of two real variables *x*, *t*. The authors of [25] have used the

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$\frac{G'}{G}$-expansion method for finding the exact solutions of the five complex nonlinear Schrödinger equations via the nonlinear Schrödinger equations, Eq. (4), the unstable Schrödinger equation

$$iu_t + u_{xx} + 2\lambda|u|^2u - 2\gamma u = 0, \tag{5}$$

and two-dimensional

$$iu_t + u_{xx} + u_{yy} + \gamma|u|^2u = 0, \tag{6}$$

and three-dimensional Schrödinger equations

$$iu_t + u_{xx} + u_{yy} + u_{zz} + \gamma|u|^2u = 0. \tag{7}$$

It is well known some traditional methods such as the finite element method (FEM), the finite volume method (FVM) and the boundary element method (BEM) are based on meshes or elements, and this is the main deficiency of these numerical techniques. In the last two decades, in order to overcome the mentioned shortcoming some methods so-called meshless methods have been proposed. There are various types of meshless techniques, for example, meshless techniques based on weak forms such as the element-free Galerkin (EFG) method [26–34], diffuse element method [35], meshless local radial point interpolation method [36–38], meshless local Petrov–Galerkin method [39–42] and including their developments; meshless techniques based on collocation techniques (strong forms) such as the meshless collocation technique based on radial basis functions (RBFs) and finally meshless techniques based on the combination of weak forms and collocation technique [43–48].

Shivanian [49,50] proposed SMRPI method which is based on meshless methods and benefits from spectral collocation ideas. In SMRPI technique, the point interpolation method with the help of those radial basis functions, which were free of shape parameter, has been proposed to construct shape functions which have Kronecker delta function property. Based on spectral methods, evaluation of high order derivatives of given differential equation is not difficult by constructing and using operational matrices.

1.1. The main aim of this paper

Our aim of this paper is the development of spectral meshless radial point interpolation to obtain the solution of two-dimensional cubic nonlinear Schrödinger equation. We will show that the SMRPI method is suitable for the treatment of the nonlinear Schrödinger equations. Also, the SMRPI has less computational complexity than the other methods that have already solved this problem, e.g. [18].

The outline of this paper is as follows: In Section 2, we obtain a time discrete scheme to the mentioned equation. In this section, we prove the unconditional stability and convergence of the time discrete scheme using the energy method. In Section 3, we introduce the spectral meshless radial point interpolation scheme briefly so that the high order operational matrices are obtained. The numerical implementation of SMRPI is given in Section 4. In Section 5, we report the numerical experiments of solving Eq. (1) for three test problems. Finally a conclusion is given in Section 6.

2. Time discretization

In this section, we discretize the time variable using forward finite difference relation for approximating the first-order derivative on time variable. If we consider Eq. (1) in point $(\mathbf{x}, t_{k+1})$, then we have

$$i\frac{u^{k+1}(\mathbf{x}) - u^k(\mathbf{x})}{\delta t} + \frac{\alpha}{2}\left(\Delta u^{k+1}(\mathbf{x}) + \Delta u^k(\mathbf{x})\right) + \beta|u^k(\mathbf{x})|u^k(\mathbf{x}) + \frac{w(\mathbf{x})}{2}\left(u^{k+1}(\mathbf{x}) + u^k(\mathbf{x})\right) + iR^{k+1} = 0, \tag{8}$$

in which $u^k(\mathbf{x}) = u(\mathbf{x}, k\delta t)$, $\delta t = t^{k+1} - t^k$ and R^{k+1} is truncation error of time discretization such that $|R^{k+1}| < C\delta t$, for a positive constant C. By simplification we have

$$i\lambda u^{k+1}(\mathbf{x}) + \alpha\Delta u^{k+1}(\mathbf{x}) + w(\mathbf{x})u^{k+1}(\mathbf{x}) = i\lambda u^k(\mathbf{x}) - \alpha\Delta u^k(\mathbf{x}) - 2\beta f(u^k) - w(\mathbf{x})u^k(\mathbf{x}) - 2iR^{k+1}, \tag{9}$$

or

$$u^{k+1}(\mathbf{x}) - i\lambda^{-1}\alpha\Delta u^{k+1}(\mathbf{x}) - i\lambda^{-1}w(\mathbf{x})u^{k+1}(\mathbf{x}) = u^k(\mathbf{x}) + i\lambda^{-1}\alpha\Delta u^k(\mathbf{x}) + 2i\lambda^{-1}\beta f(u^k) + i\lambda^{-1}w(\mathbf{x})u^k(\mathbf{x}) + R_*^{k+1}, \tag{10}$$

where $\lambda = 2/\delta t$, $f(u) = |u|^2u$ and $|R_*^{k+1}| < C_*\delta t^2$, for a positive constant C_*. By omitting the truncation local error R_*^{k+1} we obtain the following form

$$U^{k+1}(\mathbf{x}) - i\lambda^{-1}\alpha\Delta U^{k+1}(\mathbf{x}) - i\lambda^{-1}w(\mathbf{x})U^{k+1}(\mathbf{x}) = U^k(\mathbf{x}) + i\lambda^{-1}\alpha\Delta U^k(\mathbf{x}) + 2i\lambda^{-1}\beta f(U^k) + i\lambda^{-1}w(\mathbf{x})U^k(\mathbf{x}). \tag{11}$$

In the above relations $u^k(\mathbf{x})$ and $U^k(\mathbf{x})$ are exact and approximate solutions of Eq. (1), respectively.

2.1. The stability and convergence analysis

Here, we examine the stability and convergence for Eqs. (1)–(3) using the proposed time discrete scheme. To introduce the variational formulation of the Eq. (10), we define some functional spaces endowed with standard norms and inner products that will be used hereafter.

Let Ω denote a bounded and open domain in $\mathbb{R}^d$, for $d = 2$ or 3 and let dx be the Lebesgue measure on $\mathbb{R}^d$. For $p < \infty$, we denote by $L^p(\Omega)$ the space of the measurable functions $u : \Omega \to \mathbb{C}$ such that $\int_\Omega |u(x)|^p dx < \infty$ that is a Banach space with the standard norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}.$$

In particle case, let $L^2(\Omega)$ be the set of all measurable functions $u : \Omega \to \mathbb{C}$ such that $\int_\Omega |u(x)|^2 dx < \infty$. From the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, valid for all $a, b \ge 0$, we see that if $u, v \in L^2(\Omega)$ then $|u\bar{v}| \le \frac{1}{2}(|u|^2 + |v|^2)$, so that $u\bar{v} \in L^1(\Omega)$. It follows easily that the formula [51]

$$\langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}dx$$

defines an inner product on $L^2(\Omega)$. Now, above inner product induces the norm

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}.$$

Theorem 1. *Let $U^k \in L^2(\Omega)$ and we assume that the solution of Eq. (1) is analytical over $\bar{\Omega} \times [0, T]$. Also, suppose that $f(\cdot)$ has the Lipschitz condition, i.e.*

$$|f(v_1) - f(v_2)| \le \mathcal{L}|v_1 - v_2|, \quad \forall v_1, v_2 \in L^2(\Omega).$$

Then the scheme defined by (11) is unconditionally stable in $L^2(\Omega)$.

Proof. The roundoff error is to the following form

$$e^{k+1}(\mathbf{x}) - i\lambda^{-1}\alpha\Delta e^{k+1}(\mathbf{x}) - i\lambda^{-1}w(\mathbf{x})e^{k+1}(\mathbf{x}) = e^k(\mathbf{x}) + i\lambda^{-1}\alpha\Delta e^k(\mathbf{x}) + 2i\lambda^{-1}\beta(f(U^k) - f(\hat{U}^k)) + i\lambda^{-1}w(\mathbf{x})e^k(\mathbf{x}), \tag{12}$$

where $e^{k+1}(\mathbf{x}) = U^{k+1}(\mathbf{x}) - \hat{U}^{k+1}(\mathbf{x})$ whereas $U^{k+1}(\mathbf{x})$ and $\hat{U}^{k+1}(\mathbf{x})$ are exact and approximate solutions of Eq. (11), respectively. Multiplying Eq. (12) by $e^{k+1}(\mathbf{x})$ and integrating on Ω, we obtain

$$\langle e^{k+1}(\mathbf{x}), e^{k+1}(\mathbf{x})\rangle - i\lambda^{-1}\alpha\langle\Delta e^{k+1}(\mathbf{x}), e^{k+1}(\mathbf{x})\rangle - i\lambda^{-1}\langle w(\mathbf{x})e^{k+1}(\mathbf{x}), e^{k+1}(\mathbf{x})\rangle = \langle e^k(\mathbf{x}), e^{k+1}(\mathbf{x})\rangle + i\lambda^{-1}\alpha\langle\Delta e^k(\mathbf{x}), e^{k+1}(\mathbf{x})\rangle + 2i\lambda^{-1}\beta\langle(f(U^k) - f(\hat{U}^k)), e^{k+1}(\mathbf{x})\rangle + i\lambda^{-1}\langle w(\mathbf{x})e^k(\mathbf{x}), e^{k+1}(\mathbf{x})\rangle. \tag{13}$$

From $e^{k+1} \in L_0^2(\Omega) = \{u \in L^2(\Omega) : u|_{\partial\Omega} = 0\}$ and using Green's formula, we have