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# On solving free surface problems in layered soil using the method of fundamental solutions

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ABSTRACT

This paper presents the numerical solutions of free surface seepage flow in layered soil using the method of fundamental solutions (MFS). The numerical solutions are approximated by a set of fundamental solutions of the two-dimensional Laplace equation which are expressed in terms of sources located outside the domain of the problem. The unknown coefficients in the linear combination of the fundamental solutions which are accomplished by collocation imposing the boundary condition at a finite number of points can then be solved. To deal with the seepage problems of layered soil profiles, the domain decomposition method was adopted so that flux conservation and the continuity of pressure potential at the interface between two consecutive layers can be considered in the numerical model. The validity of the model is established for a number of test problems, including seepage problems in a rectangular dam, a trapezoidal dam, and an earth dam, by comparing numerical results with those from other methods. Application examples were also carried out. The results reveal that the proposed method based on the MFS has great numerical stability for solving seepage flow with nonlinear free surface in layered heterogeneous soil even with large contrasts in the hydraulic conductivity.



## 1. Introduction

Free surface seepage problems have raised much attention because it is of importance in the design of embankments, earth dams, and slope stability. The determination of the phreatic line and pressure head distributions is perhaps the crucial issue of free surface seepage analysis. In the past, researchers utilized several methods to determine the location of free surface such as Aitchison, Westbrook, Liggett and Liu, and Chen et al. [1–4]. The free surface seepage flow is governed by the Laplace equation and the solution may be carried out analytically and numerically. Due to the strong non-linearity, analyzing seepage problems with a free surface is complex while the remeshing during the iterations using conventional mesh-based numerical methods, such as the finite difference method [1,5], the boundary element method [4,6] and the finite element method [2,7–12] may lead to convergence problems. In addition, analytical solutions of the Laplace equation require several assumptions such as ideal solution domains and homogeneous material properties.

Differing from conventional mesh-based methods, the meshless method has the advantages that it does not need the mesh generation. The meshless method such as the element-free Galerkin method has been used to simulate two-dimensional seepage flow in embankments [13]. Though a background integration mesh is still used to conduct numerical integration, the nodes are independent of the mesh and can be disposed of freely. The method of fundamental solutions (MFS), first presented by Kupradze and Aleksidze in 1964 [14], is also one of the meshless methods. In the MFS, the solution is approximated by a set of fundamental solutions of the governing equation which are expressed in terms of sources located outside the domain of the problem. The unknown coefficients in the linear combination of the fundamental solutions and the final locations of the sources are determined so that the boundary conditions are satisfied in a least squares sense. The MFS needs only information of nodes on the boundary, and avoid the adjustment of mesh on the free surface during iteration. Nodes can be easily added, moved or removed, which greatly simplifies the analysis, so it is thought to be advantageous for the problem of seepage with a free surface.

The MFS has been successfully applied to a variety of problems, such as the non-linear Poisson equation [15,16], inverse problem [17–19], Stokes' problem [20], modified Helmholtz equation [21], and so on. Later, Chen and Gu [22] adopted a new formulation of singular boundary method to solve the two-dimensional potential problems. For the moving boundary problem, the MFS has been used in the bubble shape with the pressure equilibrium at the bubble boundary [23]. In 2009, Šarler [24] proposed the modified method of fundamental solutions to the potential flow problems by deriving the formulations with the single layer and the double layer fundamental solutions. Later, Perne et al. [25] developed the boundary distributed source method to solve

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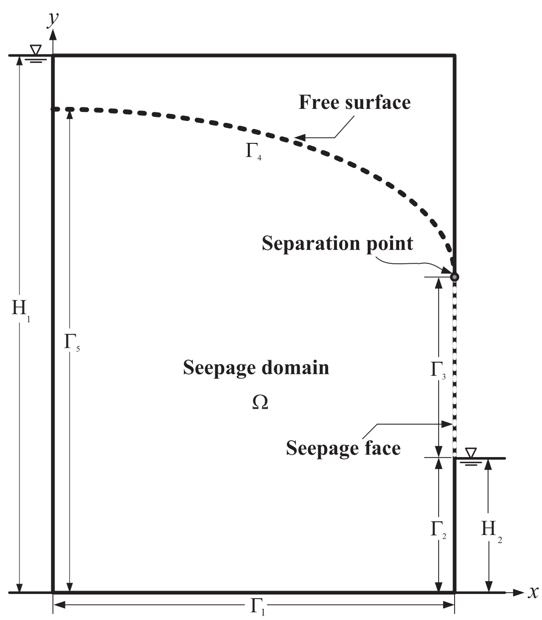


**Fig. 1.** Free surface seepage flow through a rectangular dam.

free boundary problems associated with the transport of water from the conduit to the porous matrix. Karageorghis et al. [26] proposed a moving pseudo-boundary MFS for the determination of the boundary of a void. Besides, Sincich et al. [27] presented the non-singular method of fundamental solutions for solving two-dimensional Stokes flow problems. Chaiyo et al. [28] have used the MFS to solve free boundary saturated seepage problems. Previous studies demonstrate that the MFS has been widely used to find the location of free boundary. However, the determination of the phreatic line has only been studied in the homogeneous porous medium. The study of seepage flow through dams in layered heterogeneous soil using the MFS has not been reported yet.

In this paper, the numerical solutions of seepage flow through dams in layered soil using the MFS were investigated. Free boundary is regarded as a moving boundary with the over-specified boundary conditions and the MFS is used to find the location of the free boundary. To deal with the seepage problems of layered soil profiles, the domain decomposition method (DDM) [29] was adopted so that flux conservation and the continuity of pressure potential at the interface between two consecutive layers could be considered in the numerical model. The validity of the model is established for a number of test problems, including seepage problems in a rectangular dam, a trapezoidal dam, and an earth dam, by comparing numerical results with those from other methods. Application examples were also carried out.

The remainder of this article is organized as follows. Section 2 describes the problem statement of the two-dimensional free surface seepage problem. Section 3 gives the formulation of the method of fundamental solutions to solve free surface seepage flow through dams in layered soil. In Section 4, numerical examples were conducted including rectangular dam, trapezoidal dam and earth dam. Finally, conclusions are presented in Section 5.

## 2. Problem statement

In this paper, a two-dimensional free surface seepage problem is considered in Fig. 1. The steady-state flow through a homogenous dam satisfies the Laplace governing equation as follows:

$$\Delta h(x, y) = 0 \text{ in } \Omega. \tag{1}$$

In the above equation, $h$ is the total head and $\Delta$ is the Laplace operator. Referring to Fig. 1, the boundary conditions of a rectangular dam can be presented by $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$, and $\Gamma_5$. In the $\Gamma_2$ and $\Gamma_5$, the Dirichlet boundary conditions are given as

$$h = H_2 \quad \text{on} \quad \Gamma_2, \tag{2}$$

$$h = H_1 \quad \text{on} \quad \Gamma_5. \tag{3}$$

Based on the Bernoulli equation, we neglected the velocity head and the total head or the potential can be written as

$$h = Y(x) + \frac{p}{\gamma}, \tag{4}$$

where $Y(x)$ is the elevation head, $p$ is the pressure, and $\gamma$ is the unit weight of fluid. In the $\Gamma_3$ and $\Gamma_4$, the free surface boundaries are given as over-specified boundary conditions as

$$\frac{\partial h}{\partial n} = 0,\ h = Y(x) \text{ on } \Gamma_3 \text{ and } \Gamma_4. \tag{5}$$

In the $\Gamma_1$, the no flow Neumann boundary condition to simulate the imperious boundary is given as

$$\frac{\partial h}{\partial n} = 0 \quad \text{on} \quad \Gamma_1. \tag{6}$$

Since $h = Y(x)$ is unknown in a priori which needs to be determined iteratively after the initial guess of the free surface, the MFS adopted to find the location of free boundary is expressed in the following section.

## 3. The method of fundamental solutions

Recently, the MFS has been developed and applied if the fundamental solution of the governing equation is known. The MFS is a collocation method, i.e. the boundary conditions are imposed only at a finite number of points $\mathbf{x}_k$ in which $k = 1, ..., M$ and $M$ is the number of boundary collocation points. The vector $\mathbf{x} = (x_1, x_2, ..., x_L)$ and $L$ is the dimension of Euclidean space. In the MFS, the fundamental solutions are trial functions and they satisfy the governing equation as follows:

$$\Delta F(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \tag{7}$$

where $F(\mathbf{x}, \mathbf{y})$ is the fundamental solution of the governing equation, $\Delta$ is the Laplace operator, x is the spatial coordinate which is collocated on the boundary, $\mathrm{y} = (y_1, y_2, ..., y_L)$ is the source point, and $\delta(\mathbf{x} - \mathbf{y})$ is the Dirac delta function. The solution domain is $\Omega \subset \Re^2$ with the boundary $\Gamma$. We can find an approximation solution of the two-dimensional Laplace equation as follows:

$$h(\mathbf{x}) \approx \sum_{j=1}^{N} c_j F(\mathbf{x}, \mathbf{y}_j), \tag{8}$$

where $\mathbf{x} \in \Omega$ and $\mathbf{y}_j \notin \Omega$ and $N$ is the number of source points which are placed outside the domain. The fundamental solution of Laplace equation can be obtained as

$$F(\mathbf{x}, \mathbf{y}_j) = -\frac{1}{2\pi} \ln(r_j). \tag{9}$$

It is noted that $h(\mathbf{x})$ satisfies the homogenous differential equation in the domain as a function of x and $r_j$ is defined as the distance between the boundary point and source point,

$$r_j = \left|\mathbf{x} - \mathbf{y}_j\right|. \tag{10}$$

Considering the boundary conditions, we have $h(\mathbf{x}) = g(\mathbf{x})$ and $\frac{\partial}{\partial n} h(\mathbf{x}) = f(\mathbf{x})$ where $g(\mathbf{x})$ and $f(\mathbf{x})$ represent the Dirichlet boundary condition and the Neumann boundary condition, respectively. We may select a finite number of collocation points $\mathbf{x}_k$ over the boundary such that

$$\sum_{j=1}^{N} c_j F(\mathbf{x}_k, \mathbf{y}_j) = g(\mathbf{x}_k),\ k = 1, ..., M, \tag{11}$$

where $c_j$ are the constant coefficients to be solved, and $g(\mathbf{x}_k)$ is the Dirichlet boundary condition imposed at boundary collocation points.