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An extension of the singular boundary method for solving two dimensional time fractional diffusion equations

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A R T I C L E I N F O

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A B S T R A C T

In this paper, singular boundary method (SBM) in conjunction with the dual reciprocity method (DRM) is extended to the solution of constant and variable order time fractional diffusion equations (TFDEs). In this procedure, finite difference method breaks down the time domain and reduces the time fractional diffusion equation into a sequence of boundary value problems in inhomogeneous Helmholtz-type equations. Then SBM-DRM is applied to space semi-discretization of these types of equations, in a two step process. First, DRM, which is a popular meshless method based on radial basis functions (RBFs), is applied to obtain the particular solution. After evaluating the particular solution, singular boundary method can be employed to evaluate the homogeneous solution. To consider the accuracy and efficiency of the presented method, some benchmark problems subjected to the Dirichlet and Neumann boundaries are examined on regular and irregular geometries.

1. Introduction

Anomalous diffusion phenomena which spreadingly emerged in different branches of science and engineering, such as human brain [43], contaminant transport [2] and polymer network [1], can be described and modeled well off by fractional diffusion equations. Recent studies show that constant order fractional diffusion equations cannot thoroughly describe some diffusion phenomena which have more sophisticated diffusional behavior varying according to the variation in time or space or even concentration variations [3,35]. As a result, fractional diffusion equations with the variable orders were proposed as a new achievement to deal with such issues in which the variable order fractional derivative is time dependent, spatial dependent or concentration dependent [34,36]. Due to the usage of these equations in real world problems and the difficulties to solve them with analytical methods, improving and spreading numerical methods to solve them can have a crucial role in describing real objects such as anomalous phenomena more accurately.

Time discretization with the use of finite difference method (FDM) is a common method in solving fractional diffusion equations with the constant and variable order. To simulate the space, mesh-based methods such as finite difference method [14], finite element method (FEM) [24] and boundary element method (BEM) [25], as well as kernel based methods with coordinate kernel functions such as Fourier method [4], spectral method [28] and wavelets method [12] are largely used. Although both methods are suitable for small and simple domains but they are computationally expensive for problems with high dimensions and irregular geometric domain.

In contrast to the mentioned methods, recently, meshless methods with radial basis functions on account of their dimensionality independent and avoidance troublesome mesh generation for high dimensional problems have grabbed the growing attention from a broad range of scientific computing especially in solving constant and variable order fractional differential equations [5,7,11,18,21–23]. Method of fundamental solution (MFS) is one of the RBF-kind and effectual meshless boundary collocation method. In the MFS, the main idea is to take the fundamental solutions as the function approximation and place the source points on the auxiliary boundary outside the physical boundary to regulate the singularity of the fundamental solutions [26]. Although the MFS along with being meshless, integration-free and highly accurate has drawn the growing attention in various fields of science and engineering [15,16,40,42], determining the optimal location of the auxiliary boundary, especially for problems with complex geometries, is still an open issue. This drawback makes this method less applicable for real world problems.

In contrary to MFS, SBM, which is a relatively new developed technique [8,9], figures out the troublesome placement of the fictitious boundary concerned with the traditional MFS by introducing the

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concept of the origin intensity factors which makes feasible that the source points to be positioned on the real physical boundary coinciding with the collocation points, while retaining the merits of being truly meshless, integration-free, mathematically simple and easy to program in modeling diverse problems with complicated geometries. To develop the method, subtracting and adding-back technique as well as an inverse interpolation technique are used to determine the origin intensity factors to isolate the source singularity of the fundamental solutions and their derivatives regarding Dirichlet and Neumann boundaries, respectively. Based on the numerical experiments, SBM has achieved many successes in many engineering problems such as elastic and viscoelastic wave problems [37], Stokes flow [33], acoustic [17], heat conduction [38], elasticity [20], Helmholtz [9], potential [8,39] and transient diffusion problem [10].

This paper is intended to extend singular boundary method to solve both constant and variable order time fractional diffusion equation in two-dimensional space. In this paper, first, finite difference method decomposes the time domain and approximates fractional derivative term, so that the time fractional diffusion equation is reduced to a sequence of boundary value problems in inhomogeneous Helmholtz-type equations. It is well known that, SBM similar to the other boundary type methods such as BEM and MFS is limited to solve just homogeneous equations. Therefore, in dealing with nonhomogeneous problems, we require to combine it with some other techniques. In this paper, we will combine SBM with DRM which was first introduced by Nardini [30] and has been successfully applied in combination with BEM and MFS [27,30,31] to obtain the particular solution. After evaluating the particular solution, SBM can be employed to evaluate the homogeneous solution. Finally, the real solution of the problem can be obtained by adding the homogeneous solution to the particular solution. Therefore, combining SBM and DRM, a meshless numerical technique is obtained to solve inhomogeneous and time-dependent problems.

The rest of the paper is put in order as follows. In Section 2, the basic equations of constant and variable order fractional diffusion problems are briefly introduced, followed by the complete explanation of the DRM and SBM approximation. In Section 3, we present several numerical experiments compared with the analytical solutions to illustrate the validity and efficiency of the presented method. Finally, in Section 4, some conclusions are provided upon the results reported in this study.

2. Mathematical implementation

2.1. Problem definition

In this study, a numerical investigation would be given to approximate the solution of the two-dimensional time fractional diffusion equations over a computational domain $\Omega \subset \mathbb{R}^2$ that is enclosed by a boundary Γ as follows:

$$D_t^{\alpha(t)}u(\mathbf{x},t) = A\Delta u(\mathbf{x},t) - \lambda u(\mathbf{x},t) + f(\mathbf{x},t), \quad 0<\alpha(t)<1,$$
$$\mathbf{x}\in\Omega,\ t\in(0,T), \tag{1}$$

where the symbol Δ denotes the Laplacian operator, A the diffusion coefficient and λ is the reaction coefficient. The initial condition of the problem is

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{2}$$

with the Dirichlet and Neumann boundary conditions

$$u(\mathbf{x},t) = g(\mathbf{x},t), \quad \mathbf{x}\in\Gamma_D, \quad t\in(0,T), \tag{3}$$

$$q(\mathbf{x},t) = h(\mathbf{x},t), \quad \mathbf{x}\in\Gamma_N, \quad t\in(0,T), \tag{4}$$

in which Γ_D and Γ_N stand for the parts of the boundary Ω where the Dirichlet and Neumann boundary conditions are prescribed; $f(\mathbf{x},t)$, $g(\mathbf{x},t)$, $h(\mathbf{x},t)$ and $u_0(\mathbf{x})$ are known functions, $q(\mathbf{x},t) = \frac{\partial u(\mathbf{x},t)}{\partial n}$ and n is the outward unit vector on the boundary Γ_N. Moreover, $D_t^{\alpha(t)}u(\mathbf{x},t)$ represents the variable order fractional derivative of order $\alpha(t)$ which is considered as a function taking values in the open interval (0, 1). Among the definitions which exist for variable order fractional differential equations, we adopt the definition suggested by Coimbra [13]:

$$D_t^{\alpha(t)}u(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t \frac{\partial u(\mathbf{x},\xi)}{\partial \xi}\frac{d\xi}{(t-\xi)^{\alpha(t)}} + \frac{(u(\mathbf{x},t_{0+}) - u(\mathbf{x},t_{0-}))t^{-\alpha(t)}}{\Gamma(1-\alpha(t))}, \tag{5}$$

because it only uses the integer order derivatives in the initial condition which can be simply applied in physical fields. In addition, if u is a continuous function, this definition can be viewed as the following Caputo-type definition:

$$D_t^{\alpha(t)}u(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t \frac{\partial u(\mathbf{x},\xi)}{\partial \xi}\frac{d\xi}{(t-\xi)^{\alpha(t)}}. \tag{6}$$

So, if $\alpha(t)$ is taken to be constant, this definition is reduced to the Caputo derivative definition [32].

2.2. Temporal discretization of TFDEs

Herein, the variable order time fractional derivative is approximated by the finite difference method. Let $t_k = k\tau$, $k=0,1,\dots,K$, where $\tau = \frac{T}{K}$ is the time step size and suppose $u(\mathbf{x},t)\in C(\Omega,(0,T))$. The time variable order fractional derivative $D_t^{\alpha(t)}u(\mathbf{x},t)$ at t_{k+1} can be discretized as:

$$\begin{aligned}
D_t^{\alpha(t_{k+1})}u(\mathbf{x},t_{k+1}) &= \frac{1}{\Gamma(1-\alpha(t_{k+1}))}\int_0^{(k+1)\tau}\frac{\partial u(\mathbf{x},\xi)}{\partial\xi}\frac{d\xi}{(t_{k+1}-\xi)^{\alpha(t_{k+1})}}\\
&= \frac{1}{\Gamma(1-\alpha(t_{k+1}))}\sum_{j=0}^{k}\frac{u(\mathbf{x},t_{j+1})-u(\mathbf{x},t_j)}{\tau}\\
&\quad\times\int_{j\tau}^{(j+1)\tau}\frac{1}{(t_{k+1}-\xi)^{\alpha(t_{k+1})}}d\xi\\
&= \frac{\tau^{-\alpha(t_{k+1})}}{\Gamma(2-\alpha(t_{k+1}))}\sum_{j=0}^{k} b_j^k[u(\mathbf{x},t_{k+1-j}) - u(\mathbf{x},t_{k-j})]\\
&= a^k[u(\mathbf{x},t_{k+1}) - u(\mathbf{x},t_k) + \sum_{j=1}^{k} b_j^k[u(\mathbf{x},t_{k-j+1})\\
&\quad -u(\mathbf{x},t_{k-j})]],
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
a^k &= \frac{\tau^{-\alpha(t_{k+1})}}{\Gamma(2-\alpha(t_{k+1}))}\\
b_j^k &= (j+1)^{1-\alpha(t_{k+1})} - j^{1-\alpha(t_{k+1})},\ j=0,\cdots,K-1.
\end{aligned} \tag{8}$$

We assume that $u^{k+1}(\mathbf{x})$ is an approximation of $u(\mathbf{x},t_{k+1})$. By substituting (7) into (1) and rearranging the terms, we derive the following Helmholtz type equations at each time step t_{k+1} with the unknown u^{k+1} :

$$\begin{aligned}
(\Delta-\mu^2)u^{k+1} &= F^{k+1} && k=0,\cdots,K-1\\
u^{k+1} &= h^{k+1} && on\ \Gamma_D,\\
q^{k+1} &= g^{k+1} && on\ \Gamma_N,
\end{aligned} \tag{9}$$

where $\mu = \sqrt{(\lambda+a^k)/A}$ and $F^{k+1} = \frac{1}{A}(\sum_{j=1}^{k} a^k b_j^k(u^{k-j+1} - u^{k-j}) - a^k u^k + f^{k+1})$.

Therefore, by removing the time dependence, problem (1) is turned into a series of inhomogeneous modified Helmholtz equations of the second kind.

2.3. SBM-DRM for spatial discretization

In this section, to extend the singular boundary method (SBM) to inhomogeneous Eq. (9), the dual reciprocity method (DRM) will be employed to evaluate the particular solution of the given inhomogeneous