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Micro-structured materials: Inhomogeneities and imperfect interfaces in plane micropolar elasticity, a boundary element approach

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A R T I C L E I N F O

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A B S T R A C T

In this paper we tackle the simulation of microstructured materials modelled as heterogeneous Cosserat media with both perfect and imperfect interfaces. We formulate a boundary value problem for an inclusion of one plane strain micropolar phase into another micropolar phase and reduce the problem to a system of boundary integral equations, which is subsequently solved by the boundary element method. The inclusion interface condition is assumed to be imperfect, which permits jumps in both displacements/microrotations and tractions/couple tractions, as well as a linear dependence of jumps in displacements/microrotations on continuous across the interface tractions/couple traction (model known in elasticity as *homogeneously imperfect interface*). These features can be directly incorporated into the boundary element formulation.

The BEM-results for a circular inclusion in an infinite plate are shown to be in an excellent agreement with the analytical solutions. The BEM-results for inclusions in finite plates are compared with the FEM-results obtained with FEniCS.

1. Introduction

This paper presents the first application and verification of the boundary element method to simulate the mechanical effects of inclusions with imperfect interfaces in plane micropolar elasticity.

Modern nano-technological applications such as sensors and actuators, microelecromechanical systems, electronic packaging, advanced nano-composites call for efficient approaches to model the mechanical behavior of micro and nano-structured materials. Atomistic simulations are one way forward, but these are extremely computationally expensive[1], such that multi-scale approaches are required e.g. see [2]. One approach to account for the multi-scale nature of materials is to build continuum scale constitutive theories able to reproduce the continuum behavior of such nano/micro-structured materials, see e.g. [3] for an account of continuum models of micro-structured materials. The micropolar theory is one such approach, which we use in this paper.

Micropolar (also known as Cosserat) elasticity was first introduced by the Cosserat brothers [4] and further developed by Eringen [5], Nowacki [6], Eremeyev [7] etc., and it is able to account for the rotation of individual material points (differential elements). This leads to the description of a deformed state in terms of asymmetric stress and couple stress tensors. It was shown that micropolar constitutive models, in spite of being a continuum model, are able to replicate the experimentally-observed behavior of natural or engineered materials possessing micro or nano structures [3] such as bone [8–11], fibre-reinforced composites [12–14], blocky and layered materials, such as rock and rock masses [15–17], cellular materials [18,19] and many others.

The problem of (imperfect) interfaces (also known as interphases) in Cosserat matter was scarcely addressed [20], whilst it was much more intensively modelled and simulated in the context of standard linear elasticity, with or without surface effects, see e.g. [21–26] for implementation aspects. It is however interesting to note that Cosserat materials have been themselves used to model the mechanical effects of such interphases within heterogeneous materials, as discussed in depth in recent literature [27,28].

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[1] Some estimates claim that it will be 80 years before the failure of one cubic centimeter of metal can be simulated using such approaches [1].

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Due to the rapid development of composite materials for advanced engineering applications, the problem of quantifying the effects of heterogeneities is crucially important, in particular in cases where the interfaces between the bulk/matrix and the inclusions are imperfect or carry surface energy.

The effects of heterogeneities/inhomogeneities have been studied well within the confines of Cauchy continua (classical elasticity), both analytically and numerically, starting from the classical Muskhelishvili's problem of a circular inclusion in an infinite plate [29] to the finite and boundary element analysis of multiple inclusions of various shapes, see for example [30–32] and crack/inclusion interactions, e.g. [33] and more recently [34].

In Cosserat elasticity, however, less work has been done and much remains to be understood about Cosserat-heterogeneous materials. Such efforts date back to 1976 with the work of [35]. In the 1990s significant work has been done on Cosserat-heterogeneous materials to study the effects of inclusions [36] and compute homogenized properties and their bounds and to understand their asymptotic behavior [37–39]. An interesting result of [39] is that if ℓ is the size of the Cosserat-heterogeneities, ℓ_c the Cosserat intrinsic length scales and L the size of the material sample, $\ell \approx \ell_c \ll L$ leads to a Cauchy continuum, whereas if $\ell_c \approx L$ then, the effective (homogenized) medium is better approximated by a Cosserat material.

More recently, work on Cosserat-heterogeneous materials has intensified somewhat with the work of [40], who provides analytical solutions in plane strain and [27,28] who focus on the modelling of interphases in heterogeneous materials by a non-linear Cosserat material.

A number of analytical and numerical methods have been developed to treat boundary value problems of micropolar elasticity. The finite element method remains the most common tool of numerical analysis [41–45].

Recently, the boundary element method [46] and [47] has been emerging as a powerful alternative due to its advantage in treating problems with non-smooth boundaries and infinite domains. For example, in [46] the dual boundary element method was applied to crack problems in plane strain micropolar continua.

One of the advantages of using boundary elements for inclusion problems, is the ability to incorporate the model of imperfect interfaces directly into the boundary integral formulation, keeping the linear formulation of the problem, while in the case of finite element method such interface model would make the formulation nonlinear. In this work we use the simple imperfect interface model, known as *homogeneously imperfect interface*, which is characterised by tractions and couple traction being continuous across the interface, and proportional to the jumps in displacements and out-of-plane microrotation. This model, for a circular inclusion in a plate subjected to uni-axial tension was investigated analytically in [48] with the full solution available in [49].

Another imperfect interface model, used in this work, is characterized by arbitrary jumps in both surface tractions/couple traction, as well as in displacements and out-of-plane microration. Physically, such a model allows to impose more general boundary conditions, while mathematically it brings additional advantages for the problems in infinite domain, because it enables to significantly reduce the size of the problem by transferring the boundary conditions at infinity to the boundary conditions on the inclusion interface.

In this paper we develop a system of boundary integral equations for an inclusion problem in plane micropolar and solve it by the boundary element method. We show the excellent agreement of the BEM-results with the analytical and FEM-solutions. We present the BEM-study of micropolar effects on inclusions of various shapes under various loading conditions. We demonstrate the dependence of the stress concentration factors on material parameters, including the limiting cases, when one material is nearly classical, while the second one is strongly micropolar. These parametric studies give a deeper insight into the mechanics of micropolar inhomogeneities. The developed solutions can also serve as benchmark problems for further use with other analytical and numerical methods.

The paper is organized as follows. In chapter 2 we formulate the boundary value problem of an inclusion in micropolar plane strain. In chapter 3 we derive the system of boundary integral equations. In chapter 4 we briefly outline the boundary element method procedure. Numerical results are given in chapter 5, while chapter 6 contains discussion of the results and directions of future work.

2. Mathematical formulation of an inclusion problem

According to [5], a plane strain deformation of a micropolar material is described by two in-plane displacements $u_1 = u_1(\boldsymbol{x})$, $u_2 = u_2(\boldsymbol{x})$ and one out-of-plane microrotation $\phi_3 = \phi_3(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, x_2)$, which we combine into one vector of generalized displacements: $\boldsymbol{u} = (u_1, u_2, u_3)^T$ with $u_3 = \phi_3$. In absence of body forces and couples, the equations of equilibrium for a material described by parameters $\lambda,\ \mu,\ \kappa$ and γ can be written as

$$L(\partial_x)\boldsymbol{u} = 0, \tag{1}$$

where the matrix differential operator $L(\partial_x) = L(\xi_\alpha)$ is given in [50], [51] as

$$L(\xi_\alpha) = \begin{pmatrix} (\lambda+\mu)\xi_1^2 + (\mu+\kappa)\Delta & (\lambda+\mu)\xi_1\xi_2 & \kappa\xi_2 \\ (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_2^2 + (\mu+\kappa)\Delta & -\kappa\xi_1 \\ -\kappa\xi_2 & \kappa\xi_1 & \gamma\Delta - 2\kappa \end{pmatrix}, \tag{2}$$

with $\xi_\alpha = \partial/\partial x_\alpha$ and $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 = \xi_1^2 + \xi_2^2$.

Two tractions $t_1 = t_1(\boldsymbol{x})$, $t_2 = t_2(\boldsymbol{x})$ and one couple-traction $t_3 = t_3(\boldsymbol{x})$, defined on a boundary with normal $\boldsymbol{n} = (n_1, n_2)^T$, are also combined into vector $\boldsymbol{t} = (t_1, t_2, t_3)^T$. By the standard definition

$$t_\alpha = \sigma_{\beta\alpha} n_\beta, \quad t_3 = m_{\beta 3} n_\beta, \quad \alpha, \beta = 1, 2. \tag{3}$$

where σ_{11}, σ_{12}, σ_{21}, σ_{22} are components of the asymmetric micropolar stress tensor and m_{13}, m_{23} are the couple-stresses.

Together with $L(\xi_\alpha)$ the boundary stress operator $T(\partial_x) = T(\xi_\alpha)$ is considered [50], which is defined by the following equation:

$$T(\xi_\alpha) = \begin{pmatrix} (\lambda+2\mu+\kappa)\xi_1 n_1 + (\kappa+\mu)\xi_2 n_2 & \lambda\xi_2 n_1 + \mu\xi_1 n_2 & \kappa n_2 \\ \mu\xi_2 n_1 + \lambda\xi_1 n_2 & (\mu+\kappa)\xi_1 n_1 + (\lambda+2\mu+\kappa)\xi_2 n_2 & -\kappa n_1 \\ 0 & 0 & \gamma\xi_\alpha n_\alpha \end{pmatrix} \tag{4}$$

Operator $T(\partial_x)$ is defined according to the stress strain relations and the constitutive equations, as given in [5] in such a way that

$$\boldsymbol{t} = T(\partial_x)\boldsymbol{u}. \tag{5}$$

Together with constants $\lambda,\ \mu,\ \gamma,\ \kappa$, we use engineering constants: G (shear modulus), ν (Poisson's ratio), ℓ (characteristic length) and N (coupling number), defined in [8].

We consider a bounded inclusion occupying the domain S^i with the boundary ∂S^i and inner normal $\vec{n}$ as shown in Fig. 1. The inclusion is made of homogeneous and isotropic micropolar material with elastic constants $\lambda^i,\ \mu^i,\ \kappa^i,\ \gamma^i$. The matrix, which occupies domain S^e is also homogeneous and isotropic micropolar material with elastic constants λ^e, $\mu^e,\ \kappa^e,\ \gamma^e$. The engineering material parameters, describing the inclusion or the matrix are denoted as $G^i,\ \nu^i,\ \ell^i,\ N^i$ or $G^e,\ \nu^e,\ \ell^e,\ N^e$ respectively.

Let $L^i(\partial_x)$ and $L^e(\partial_x)$ be the operator $L(\partial_x)$ with constants $\lambda^i,\ \mu^i,\ \kappa^i,\ \gamma^i$ and $\lambda^e,\ \mu^e,\ \kappa^e,\ \gamma^e$ respectively. The boundary stress operators $T^i(\partial_x)$ and $T^e(\partial_x)$ are defined analogously. The displacement vector in domain S^i is denoted as $\boldsymbol{u}^i$, in domain S^e as $\boldsymbol{u}^e$. The boundary tractions are defined as

$$\boldsymbol{t}^i = T^i(\partial_x)\boldsymbol{u}^i, \quad \boldsymbol{t}^e = T^e(\partial_x)\boldsymbol{u}^e. \tag{6}$$