Engineering Analysis with Boundary Elements 83 (2017) 217–228

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Stability and accuracy improvement for explicit formulation of time domain acoustic problems

Eric Li[b], ZC He[a,*]

[a] State Key Laboratory of Advanced Design and Manufacturing for Vehicle Body, Hunan University, Changsha 410082, PR China
[b] Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Shatin, NT, Hong Kong, China

ARTICLE INFO

Keywords:
Acoustic
Stability
Dispersion error
Softened stiffness
Integration point

ABSTRACT

Due to the "wave velocity error" between the discretized model and continuum systems, the accuracy of numerical results using standard finite element method (FEM) in time domain acoustic problems is unsatisfactory with increasing frequency. Such wave velocity error is strongly related to the balance between the "stiffness" and "mass" of discretized systems. By adjusting the location of integration point of mass matrix, the redistribution of the mass is able to "tune" the balance between the stiffness and mass. Thus, the wave velocity error can be minimized in time domain acoustic problems with a balance system. On the other hand, it is found that the stability of discretized model of time domain acoustic problems can be improved by the softened stiffness. Furthermore, it is found that the balance between the smoothed stiffness and mass can also be achieved with the tuning of integration point r in the mass matrix.

1. Introduction

Recently, with increasing customer demands on the comfort of aircrafts cabin and vehicle passenger compartments, the need for the numerical solutions governed by the Helmholtz equation in the design of acoustic systems is becoming more and more important. Currently, most of the numerical solutions for such designs are still based on the standard finite element method (FEM) [1] and boundary element method (BEM) [2] using the commercially available software packages. However, the numerical solutions using FEM and BEM will deteriorate in the middle frequency range due to the dispersion error [3–5], which is the so-called numerical "pollution" of discretization errors in the amplitude and phase (dispersion error). The refinement of mesh is a possible way to improve the numerical solutions of FEM and BEM. However, the computational cost will increase significantly especially for large scale 3D problems using sufficiently fine mesh. In order to enhance computational efficiency of acoustic problems, many numerical methods including the Galerkin/least-squares finite element method (GLS) [6,7], the Galerkin-gradient/least-squares method (GGLS) [8], the generalized finite element method (GFEM) [9,10] and the residual-free finite element method (RFFEM) [11] have been proposed. However, very few of these methods may eliminate the pollution errors in general 2D and 3D acoustic problems [12].

On the other hand, the appropriate element types for the discretization of physical structure in the industry is also very important. In general, the low order elements such as three-node triangular (T3) in 2D and four-node tetrahedral (T4) elements in 3D are preferred by engineers due to the easy creation of mesh for any complex geometry. In addition, the low-order elements allow very convenient h-type mesh refinement adaptation, leading to easier automation in modeling and simulation for domains with complicated geometries [13]. However, the low-order triangular or tetrahedral elements in the FEM model exhibit overly-stiff behavior, leading to the loss of balance between the stiffness and mass matrices. Thus, the results for acoustic problems using lower order of FEM especially in the mid-frequency range [14] are relatively poor. Although the higher order elements of FEM can give higher convergence rate for static and dynamic problems [15], the numerical results for the eigenvalues of higher modes (higher frequencies) can deteriorate drastically in the analysis of dynamics problems.

Recently, Liu's group has found that the generalized gradient smoothing operations [16–18] is able to effectively soften overly-stiff of stiffness for traditional FEM model, known as the softening effects. By using the simple triangular and tetrahedral mesh, various types of smoothed finite element methods (S-FEM) with different levels of smoothing effects [19–27] have been developed. Compared with standard FEM, S-FEM is able to give ultra-accurate solution, high convergence rate and better computational efficiency in many engineering areas [20, 28–35]. In particular, the past works have demonstrated that the edge-based smoothed finite element (ES-FEM) with easy implementation provides a proper "softness" to the model [36–40], which gives

* Corresponding author.
E-mail address: hezhicheng815@hnu.edu.cn (Z. He).

http://dx.doi.org/10.1016/j.enganabound.2017.07.014
Received 22 May 2017; Received in revised form 17 July 2017; Accepted 17 July 2017
0955-7997/

much more accurate results in frequency domain of acoustic problems compared with standard FEM [14].

In the simulation of time or frequency domain acoustic problems, the mass matrix is involved in the discretized model. Therefore, an alternative approach to improve the low-order element is to modify the mass matrix to balance the discretized model between stiffness and mass matrix. Marfurt [41] found that the FEM using a weighted average of lumped and consistent and masses achieved the very accurate results. Furthermore, the modified integration rules were proposed by Murthy to compute the stiffness and mass matrix using quadrilateral mesh for acoustic problems [42,43], leading to very accurate numerical results with second-order accuracy compared with the traditional integration. In addition, Idesman [44] also modified the integration rules to compute the mass and stiffness matrices to achieve a good balance between them in the analysis of elastodynamics problems, which causes the reduction of numerical dispersion significantly. All these numerical methods were performed based on quadratic elements and the tuning is done for both stiffness and mass matrices. Following this, we proposed a mass-redistributed finite element method (MR-FEM) for acoustic problems using triangular and tetrahedral elements in the frequency domain [45,46]. The key concept of MR-FEM is just to adjust the integration point in the mass matrix to tune the balance of mass and stiffness using linear triangular and tetrahedral elements [45–50].

In this work, the optimal integration point in the generalized mass matrix created by MR-FEM using linear triangular element is furthered investigated in time domain acoustic problems. Different from frequency domain, the stability and accuracy of numerical solutions depend on both time step and spatial discretization in the simulation of time domain acoustic problems. In order to narrow down this research, time integration of central difference method with combination of MR-FEM is focused. Based on the dispersion error analysis, it is found that the dispersion error can be minimized with the optimal integration point r in the mass matrix. Compared with traditional consistent and lumped mass matrix, MR-FEM can provide much more accurate results. In addition, the balance of smoothed stiffness created by ES-FEM and mass is also studied in this work. It is found that ES-FEM can give the best solutions when the optimal integration $r=0$ or $r=2/3$ in the mass matrix (corresponding to consistent mass) is adopted. Furthermore, the stability of time domain acoustic problems is investigated in this work. It is found that critical time step in the ES-FEM and FEM model is proportional to integration points of mass matrix r as $r>1/3$. More importantly, for the first time, it is observed that ES-FEM behaves more stable compared with FEM in the transient acoustic problems as the same integration point r is employed in the mass matrix. This is due to the softened effect in the stiffness caused by ES-FEM. Our important study in this work from mathematical perspective and numerical experiments has revealed two key facts in the simulation of transient acoustic problems: the stability of transient acoustic problems can be improved by the smoothed stiffness; the accuracy of numerical solutions can be improved significantly by adjusting the integration point in the mass matrix.

The paper is structured as follows: the mathematical model of time domain acoustic problems together with the formulation of MR-FEM is described in Section 2. The stability of time domain acoustic problems is presented in Section 3. Section 4 illustrates the theoretical study of minimization of wave velocity error in the time domain acoustic problems. A number of numerical examples are studied in detail to analyze the accuracy and stability of time domain acoustic problems in Section 5. Finally the conclusions from the numerical solutions are made in Section 6.

2. Mathematical model of time domain acoustic problem

2.1. Discretized of equation

Consider a 2D time domain acoustic problem, and the domain Ω with boundary Γ can be decomposed into three portions Γ_D, Γ_N and Γ_A, where $\Gamma=\Gamma_D\cup\Gamma_N\cup\Gamma_A$. The governing equation for the time domain acoustic wave can be expressed as follows:

$$\Delta p-\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}=0 \tag{1}$$

where p represents the acoustic pressure, c is the speed of sound traveling in the medium, Δ stands for the Laplace operator and t is time.

The Dirichlet, Neumann and Robin boundary conditions on Γ_D, Γ_N and Γ_A can be given as follows:

$$p=p_D \quad \text{on}\,\Gamma_1 \quad \text{Dirichlet condition} \tag{2}$$

$$\nabla p\cdot n=-\rho\dot{v}_n \quad \text{on}\,\Gamma_2 \quad \text{Neumann condition} \tag{3}$$

$$\nabla p\cdot n=-\rho A_n\frac{\partial p}{\partial t} \quad \text{on}\,\Gamma_3 \quad \text{Robin condition} \tag{4}$$

where v_n, ρ and A_n represent the normal velocity on the boundary Γ_N, the density of medium and admittance of the structural damping on boundary Γ_A, respectively.

The acoustic weak form of system equations can be formulated by discretizing the wave equation Eq. (1) using FEM, and the integration is performed based on element.

$$\int_\Omega(\nabla\delta p)\cdot\nabla p\mathrm{d}\Omega+\frac{1}{c^2}\int_\Omega\delta p\frac{\partial^2 p}{\partial t^2}\mathrm{d}\Omega+\rho\int_{\Gamma_2}\delta p\dot{v}_n\mathrm{d}\Gamma+\rho A_n\int_{\Gamma_3}\delta p\frac{\partial p}{\partial t}\mathrm{d}\Gamma=0 \tag{5}$$

The pressure can be expressed in the approximate form.

$$p=\sum_{i=1}^{m}\mathbf{N}_i p_i=\mathbf{N}\mathbf{p} \tag{6}$$

$$\delta p=\sum_{i=1}^{m}\mathbf{N}_i p_i=\mathbf{N}\delta\mathbf{p} \tag{7}$$

where m is the number of nodal variables of the element, $\mathbf{N}_i$ are FEM shape functions and p_i is the unknown nodal pressure. By substituting the approximation as indicated in Eqs. (6) and (7) into the Galerkin weak form and invoking the arbitrariness of virtual node pressure, the discretized system equation can then be obtained and written in the following matrix form:

$$[\mathbf{M}]\{\ddot{\mathbf{p}}\}+[\mathbf{C}]\{\dot{\mathbf{p}}\}+[\mathbf{K}]\{\mathbf{p}\}=\{\mathbf{F}\} \tag{8}$$

where

$$\mathbf{K}=\int_\Omega\nabla\mathbf{N}^{\mathrm{T}}\nabla\mathbf{N}\mathrm{d}\Omega \quad \text{The acoustical stiffness matrix} \tag{9}$$

$$\mathrm{M}=\frac{1}{c^2}\int_\Omega\mathbf{N}^{\mathrm{T}}\mathbf{N}\mathrm{d}\Omega \quad \text{The acoustical mass matrix} \tag{10}$$

$$\mathbf{C}=\rho A_n\int_{\Gamma_3}\mathbf{N}^{\mathrm{T}}\mathbf{N}\mathrm{d}\Gamma \quad \text{The acoustical damping matrix} \tag{11}$$

$$\mathbf{F}=-\rho\int_{\Gamma_2}\mathbf{N}^{\mathrm{T}}\dot{v}_n\mathrm{d}\Gamma \quad \text{The vector of nodal acoustic forces} \tag{12}$$

$$\{\mathbf{p}\}^T=\{p_1,p_2,\ldots,p_n\} \quad \text{Nodal acoustic pressure in the time domain} \tag{13}$$

In this work, the edge-based smoothed finite element method (ES-FEM) [36] with optimal integration point r in the generalized mass matrix created by MR-FEM to solve the transient acoustic problems is also discussed. In the ES-FEM, by introducing the gradient smoothing technique based on the edges of elements as shown in Fig. 1, the gradient component $\nabla\mathbf{N}$ in Eq. (9) is replaced by the smoothed item $\nabla\overline{\mathbf{N}}$.

$$\mathbf{K}_{\mathrm{ES}}=\int_\Omega\nabla\overline{\mathbf{N}}^{\mathrm{T}}\nabla\overline{\mathbf{N}}\mathrm{d}\Omega \quad \text{The smoothed stiffness matrix} \tag{14}$$