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A high-order three-dimensional numerical manifold method enriched with derivative degrees of freedom

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ABSTRACT

A three-dimensional (3D) high-order numerical manifold method (NMM) is developed based on the partition of unity method (PUM). We enrich the high-order NMM by introducing the derivative degrees of freedom associated with explicit physical significance. The global displacement in the formulation is approximated by a second-order approximation for the local displacement in conjunction with a first-order weight function. This not only helps the high-order NMM effectively avoid the problem of linear dependence that is frequently encountered in the PUM, but also renders the stress or strain at the star points continuous for the high-order NMM without the necessity of further smoothing operation. The effectiveness and robustness of the proposed new high-order NMM are demonstrated by several typical examples. Future potential developments and applications of the method are discussed.

1. Introduction

The numerical manifold method (NMM) [1] has received increasing attention in a wide range of engineering research areas, including fracture mechanics [2–8], fluid dynamics [9,10], seepage flow [11,12], the fourth order problems [13], the functionally graded materials [14], and isogeometric analysis [15]. Typically based on the first-order partition of unity (PU) [16], the NMM has recently been extended to higher orders, including the second [17] and the third order [18] developments with the addition of mathematical patches (MP) to cover a manifold element (ME), and the higher-order extension in [19] based on raising the order of local approximation. The various formulations of high-order NMM commonly suffer a serious issue of linear dependence (LD) which may further cause the notorious rank deficiency (RD) issue of the global stiffness matrix. To address this, a new algorithm has been developed [20] and further extended [21] to predict the RD. A dual local approximation scheme has also been introduced [22], and some strategies have been suggested in [23] to suppress this phenomenon. More recently, a two-dimensional (2D) high-order NMM with derivative degrees of freedom has been proposed by the authors [24] which may help to avoid the issue of linear dependence. Other latest developments of NMM encompass extensions based on an explicit formulation [25] and involving of strain-rotation decomposition to resolve large deformation and large rotation issues [26–28].

While 2D simplifications may be applicable to a number of cases, most real engineering problems are three-dimensional (3D). And as the enhanced version of FEM, 3D extended finite element method (3D-XFEM) was developed [29]. 3D generalized finite element method (3D-GFEM) was also proposed for investigating some 3D structural mechanics problems [30]. In addition, 3D mesh-free method [31] has also achieved some certain progress. It is hence desirable a full 3D NMM could be developed for practical application. There are two major challenges related to the 3D extension from a 2D NMM. (1) Choice of mesh. It may seem to be straightforward to formulate the 3D NMM using 4-node tetrahedral meshes [32,33]. However, for a practical problem, regular tetrahedrons may not completely fill a 3D space. Hence, the tetrahedral meshes at domain boundaries have to be subdivided and replaced by refined tetrahedrons [34,35]. Other mesh type can also be used, for example, hexahedral meshes have recently been employed in developing a new augmented NMM with flat-top PU by He et al. [36], which helps to avoid the issue of linear dependence in the NMM. A fault-cutting algorithm based on hexahedral meshes has also been developed [37]. (2) Physical significance of the undetermined coefficients. In most 2D NMM, the undetermined coefficients, or so-called as generalized degrees of freedom (DOFs) that are associated with the basis of the local approximation, do not possess concrete physical meanings. A recent study [23] has borrowed the concept of DOFs used in discontinuous deformation analysis (DDA) [38,39] for improvements. Similar ideas have been further used for a high-order NMM [24].

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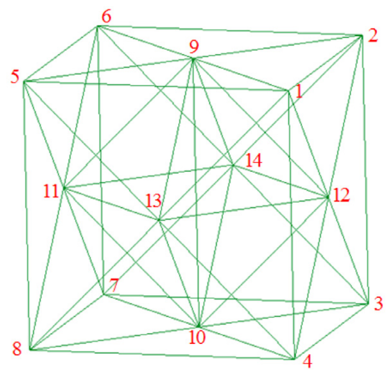

Twenty four tetrahedrons expressed by node number:

1: 1-12-4-13	9: 7-8-10-11	17: 9-11-13-5
2: 1-13-5-9	10: 5-8-11-13	18: 9-14-11-6
3: 1-2-12-9	11: 4-10-8-13	19: 10-11-14-7
4: 2-3-12-14	12: 3-4-12-10	20: 10-13-11-8
5: 2-9-6-14	13: 9-13-12-1	21: 11-13-14-9
6: 5-6-9-11	14: 9-12-14-2	22: 12-14-13-9
7: 6-7-14-11	15: 10-14-12-3	23: 11-14-13-10

Fig. 1. A hexahedron composed by 24 tetrahedrons.

This study aims at developing a high-order 3D NMM based on the tetrahedral meshes. A new local approximation is proposed to construct the global approximation based on the principle of PU. The DOFs with attributed physical meaning are incorporated into the high-order 3D-NMM, and its linear independence is verified by counting the number of zero eigenvalues of the global stiffness matrix [40,41]. The new local approximation for the 3D-NMM leads to a continuous stress field at the star point, hence avoiding the necessity of extra smoothing operation on the stress field.

2. Brief introduction of the NMM

In a 3D space, an arbitrary shape of problem domain can be filled by a mesh of hexahedrons. Each hexahedral element may be further subdivided into a number of tetrahedrons. Fig. 1 shows a hexahedron consisting of 24 tetrahedrons. A problem domain is referred to as the physical cover (PC) in the NMM. Fig. 2 shows a PC outlined by the black solid line with 120 tetrahedrons marked by the gray dotted line wherein we will focus on the specific tetrahedron manifold element (ME) 1234 highlighted by the red solid line. In the tetrahedral mesh, all tetrahedrons share the same node form a mathematical patch (MP) and the communal node is called the star. All these MPs forms a collective named mathematical cover (MC). It should be pointed out that the MPs can be an arbitrary geometry polyhedron, sphere, and ellipsoid, and among others. In Fig. 2, each MP is a polyhedron, see, *e.g.*, MP_1, MP_2, MP_3, and MP_4 associated with the tetrahedron 1234. Cutting the PC with the MPs generates the physical patches (PPs). For instance, MP_1, MP_2, MP_3, and MP_4 are cut by the resolution domain to form PP_1, PP_2, PP_3, and PP_4. The intersection of these four PPs then creates the ME 1234, as shown in Fig. 3. It is postulated that the global approximation defined over every ME is related to its correspounding 4 PPs. One can refer to Refs. [1,6,23,42] for more detailed formulation and description of the NMM.

3. Local and global approximation

The constant, ordinary power or trigonometric series can serve as the basis of function of NMM's local displacement approximation defined over each PP. If the power series is employed, for the kth PP it reads

$$\mathbf{u}_k(x,y,z)=\begin{Bmatrix}u_k(x,y,z)\\ v_k(x,y,z)\\ w_k(x,y,z)\end{Bmatrix}=\sum_{j=1}^{m}\begin{pmatrix}b_{kj}(x,y,z)&0&0\\0&b_{kj}(x,y,z)&0\\0&0&b_{kj}(x,y,z)\end{pmatrix}\begin{pmatrix}d_{k3j-2}\\d_{k3j-1}\\d_{k3j}\end{pmatrix}\tag{1}$$

where $b_{k,j}(x,y,z)$ is the basis function of a local displacement approximation and m is the number of $b_{k,j}(x,y,z)$. Assume that the number of PPs is n, and there are $3m$ unknowns in each PP, namely

$$D_k=\begin{pmatrix}d_{k1}\\d_{k2}\\\cdots\\d_{k3m-2}\\d_{k3m}\end{pmatrix},\quad k=1,2,\ldots,n\tag{2}$$

where D_k is a basic unknown coefficient vector and has no apparent physical meaning, being referred to as generalized degrees of freedom. Nevertheless, the degrees of freedom of 2D-DDA have been endowed with the physical meaning and have been used to construct the local displacement approximation of NMM [24]. Following the core idea of Ref. [24], in this study, we will adopt the degrees of freedom of 3D-DDA [43] to establish the local approximation. Namely, Eq. (1) can be rewritten as

$$\mathbf{u}_k=\mathrm{T}^k\mathbf{d}_k\tag{3}$$

where

$$\mathbf{T}^k=\begin{bmatrix}N_k&0&0&N_{kx}&0&0&0&\frac{N_{kz}}{2}&\frac{N_{ky}}{2}&0&N_{kz}&-N_{ky}\\0&N_k&0&0&N_{ky}&0&\frac{N_{kz}}{2}&0&\frac{N_{kx}}{2}&-N_{kz}&0&N_{kx}\\0&0&N_k&0&0&N_{kz}&\frac{N_{ky}}{2}&\frac{N_{kx}}{2}&0&N_{ky}&-N_{kx}&0\end{bmatrix}\tag{4}$$

and

$$\mathbf{d}_k=\left\{u^k\quad v^k\quad w^k\quad \varepsilon_x^k\quad \varepsilon_y^k\quad \varepsilon_z^k\quad \gamma_{yz}^k\quad \gamma_{zx}^k\quad \gamma_{xy}^k\quad \omega_x^k\quad \omega_y^k\quad \omega_z^k\right\}^{\mathrm{T}}\tag{5}$$

where ε_x^k, ε_y^k, ε_z^k, γ_{yz}^k, and γ_{xy}^k are the strain components at star (x_k,y_k,z_k). Moreover, ω_x^k, ω_y^k, and ω_z^k are the rotational angle of any infinitesimal vector passing the same star around the x-, y- and z-axis, respectively. Apparently, the basic unknown vector, $\mathbf{d}_k$, has clear physical meanings. In conjunction with Eqs. (4) and (5), after some mathematical manipulations, Eq. (3) can be rewritten as

$$\mathbf{u}_k(x,y,z)=\begin{Bmatrix}u_k(x,y,z)\\v_k(x,y,z)\\w_k(x,y,z)\end{Bmatrix}=\begin{pmatrix}N_ku^k+N_{kx}\varepsilon_x^k+\frac{N_{kz}}{2}\gamma_{zx}^k+\frac{N_{ky}}{2}\gamma_{xy}^k+N_{kz}\omega_y^k-N_{ky}\omega_z^k\\N_kv^k+N_{ky}\varepsilon_y^k+\frac{N_{kz}}{2}\gamma_{yz}^k+\frac{N_{kx}}{2}\gamma_{xy}^k-N_{kz}\omega_y^k+N_{kx}\omega_z^k\\N_kw^k+N_{kz}\varepsilon_z^k+\frac{N_{ky}}{2}\gamma_{yz}^k+\frac{N_{kx}}{2}\gamma_{zx}^k+N_{ky}\omega_x^k-N_{kx}\omega_y^k\end{pmatrix}\tag{6}$$

The shape functions defined on a tetrahedral mesh *ijmn* (see Fig. 4) is introduced

$$L_k=\frac{\tilde{a}_k+\tilde{b}_kx+\tilde{c}_ky+\tilde{d}_kz}{6V}=a_k+b_kx+c_ky+d_kz,\quad k=i,j,m,n\tag{7}$$