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Efficient boundary integral solution for acoustic wave scattering by irregular surfaces

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ABSTRACT

The left–right operator splitting method is studied for the efficient calculation of acoustic fields scattered by arbitrary rough surfaces. Here, the governing boundary integral is written as a sum of left- and right-going components, and the solution expressed as an iterative series, expanding about the predominant direction of propagation. Calculation of each term is computationally inexpensive both in time and memory, and the field is often accurately captured using one or two terms. The convergence and accuracy are examined by comparison with exact solution for smaller problems, and a series of much larger problems are tackled. The method is also immediately applicable to other scatterers such as waveguides, of which examples are given.

1. Introduction

The calculation of acoustic scattering by extended rough surfaces remains a challenging problem both theoretically and computationally (e.g. [1–6]) especially in the presence of strong multiple scattering. This becomes acute at low grazing angles, where multiple scattering occurs for very slight roughness. Boundary integral methods [7,8] are flexible and often used for such problems but can be computationally intensive and scale badly with increasing wavenumber. Much effort has therefore been devoted to this aspect, where possible exploiting properties of the scattering regime. For forward scattering in 2-dimensions, for example, provided roughness length-scales are large, the 'parabolic integral equation method' can be applied [9,10]. For electromagnetic problems, also formulated using boundary integrals, the methods of ordered multiple interactions and left–right splitting in both 2-d and 3-d [11,12] have been developed: here, the scattered field is expressed as an iterative series of terms of increasing orders of multiple scattering, as described below. Approaches using conjugate gradient solutions combined with fast multilevel multipole are also receiving much attention. An important exception which overcomes the dependence of computational expense on wavenumber is [13], which has been applied to surfaces with piecewise constant impedance data or scattering in 2-d by convex polygons.

A versatile recursive technique known as Multiple Sweep Method of Moments was developed and analysed in [14,15] where it was compared with Method of Ordered Multiple Interactions. This technique was shown to tackle 'composite' problems for which the above method diverges such as for a ship on a rough sea surface. Other iterative solutions have been studied in [16]. In addition, theoretical results are available in various limiting regimes (e.g. perturbation theory for small surface heights, $k\sigma \ll 1$ including periodic surfaces [17–19], Kirchhoff approximation [20,21], or the small slope approximation [22] which is accurate over a wider range of scattering angles than both of these). For arbitrary finite rough surfaces, however, validation is more difficult, and such results are therefore scarce.

In this paper the Left–Right Splitting method is developed and applied to the problem of acoustic scattering in three dimensions by randomly rough surfaces. For relatively small surfaces the results are validated by comparison with numerical solution of the full boundary integral equation. The principal aims are to validate the approach; to examine its robustness and convergence as the angle of incidence changes; and to consider further approximations which may reduce the computation time. The approach is applicable to a wide range of interior and exterior scattering problems, and we give examples for acoustic propagation in a varying duct, in addition to scattering from large rough surfaces.

The mathematical principles of the method are the same as for the two-dimensional problem [23] although implementation is considerably more complicated: the unknown field ψ on the surface is expressed as the solution to the Helmholtz integral equation, with the integration taken over the rough surface. This may be written formally as $A\psi = \psi_{\mathrm{inc}}$, where ψ_{inc} is the incident field impinging (say) from the left, so that we require $\psi = A^{-1}\psi_{\mathrm{inc}}$. The region of integration is split into two, to the left and right of the point of observation, allowing A to be written as the sum of 'left' and 'right' components, say $(L + R)\psi = \psi_{\mathrm{inc}}$. Roughly

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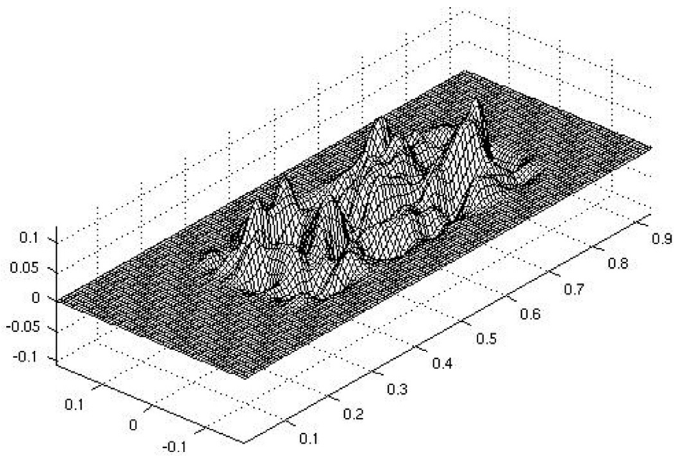

Fig. 1. Example rough surface.

speaking L represents surface interactions due to scattering from the left, and R the residual scattering from the right. The inverse of A can formally be expressed as a series

$$A^{-1} = L^{-1} - L^{-1}RL^{-1} + \dots. \tag{1}$$

Discretization of the integral equation yields a block matrix equation, in which L is the lower triangular part of the block matrix A (including the diagonal) and R is the upper triangular part. Under the assumption that most energy is right-going, L is the dominant part of A, and the series can be truncated to provide an approximation for ψ. This approach has several advantages. In terms of wavelength λ, evaluation of each term scales with the fourth rather than the sixth power of λ required for A^{-1}; subsequent terms (of which typically only the first one or two are needed) have the same computational cost. With further approximations this can be reduced to λ^3. However, this operation count is only part of the story, because the low complexity and memory requirement allow very large problems to be tackled without such additional approximation. In addition the algorithm lends itself well to parallelization, and the speed scales approximately linearly with the number of processors.

In Section 2 the governing equations and left–right splitting approximation are formulated. The numerical details and main results are shown in Section 3.

2. Formulation of equations

Consider a 3-dimensional medium with horizontal axes *x, y* and vertical axis z directed upwards, and let k be the wavenumber. Let $S = s(x, y)$ be a 2-dimensional rough surface, varying about the plane $z = 0$, which is continuous and differentiable as a function of *x, y* (see Fig. 1). (Arbitrary scatterers can also be treated by the methods shown here; examples will be given later.) Consider a time-harmonic acoustic wave ψ, obeying the wave equation $(\nabla^2 + k^2)\psi = 0$ in the region $z > s(x, y)$, resulting from an incident wave $\psi_{\rm inc}$ at a small grazing angle θ to the horizontal plane. This may for example be a plane wave or a finite beam. The axes can be chosen so that the principal direction of propagation is at a small angle to the (*x, z*) plane.

We will treat the Neumann boundary condition, i.e. an acoustically hard surface. The derivation for the Dirichlet condition is similar. The starting point for this treatment is the boundary integral formulation [4,7,8]. Thus

$$\frac{\partial \psi}{\partial \mathbf{n}} = 0 \tag{2}$$

where $\mathbf{n}$ is the outward normal (i.e. directed out of the region $z > s(x, y)$). The free space Green's function is given by

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{3}$$

The field at a point $\mathbf{r}$ in the medium is related to the surface field by the boundary integral

$$\psi_{\rm inc}(\mathbf{r}) = \psi(\mathbf{r}) - \int_S \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} \psi(\mathbf{r}') d\mathbf{r}' \tag{4}$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', s(x', y'))$, say, and taking the limit as $\mathbf{r} \to \mathbf{r}_s$ gives

$$\psi_{\rm inc}(\mathbf{r}_s) = \psi(\mathbf{r}_s) - \int_S \frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial n} \psi(\mathbf{r}') d\mathbf{r}' \tag{5}$$

where now $\mathbf{r}_s = (x, y, s(x, y))$. The integrand is singular at the point $\mathbf{r}' = \mathbf{r}_s$, and we must take care to interpret this integral as the limit of the integral in Eq. (4) as $\mathbf{r} \to \mathbf{r}_s$.

In order to treat the equation numerically it is convenient to write the integration with respect to *x, y*, so that Eq. (5) becomes

$$\psi_{inc}(\mathbf{r}_s) = \psi(\mathbf{r}_s) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial n} \psi(\mathbf{r}')\, \gamma(\mathbf{r}') dx' dy' \tag{6}$$

where (with very slight abuse of notation)

$$\gamma(\mathbf{r}') = \sqrt{1 + \left(\frac{\partial s}{\partial x'}\right)^2 + \left(\frac{\partial s}{\partial y'}\right)^2}. \tag{7}$$

and the expression under the square root is evaluated at $\mathbf{r}'$.

2.1. Formal solution and splitting series

The method of solution is analogous to that applied to the electromagnetic problem in 2-d or 3-d [23,24]. The governing integral equation (6) is expressed in terms of right- and left-going operators L and R with respect to the x-direction:

$$\psi_{\rm inc}(\mathbf{r}_s) = A\psi \equiv (L + R)\psi \tag{8}$$

where L and R are defined (for an L^2 function *f*) by

$$Lf(\mathbf{r}) = f - \int_{-\infty}^{\infty}\int_{-\infty}^{x} \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} f(\mathbf{r}')\, \gamma(\mathbf{r}')\, dx'\, dy', \tag{9}$$

$$Rf(\mathbf{r}) = -\int_{-\infty}^{\infty}\int_{x}^{\infty} \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} f(\mathbf{r}')\, \gamma(\mathbf{r}')\, dx'\, dy' \tag{10}$$

and $\mathbf{r} = (x, y, z)$, $\mathbf{r}' = (x', y', s(x', y'))$. (For notational conveneince L is interpreted to include the contribution from the singularity arising in (5) when $\mathbf{r} \to \mathbf{r}'$.)

The region of integration is thus split into two with respect to x, and the solution of Eq. (8) can be expanded as a series, given by

$$\psi = (L + R)^{-1}\psi_{inc} = \left[L^{-1} - L^{-1}RL^{-1} + \dots\right]\psi_{\rm inc}. \tag{11}$$

The key observation is that at fairly low grazing angles the effect of $\mathbf{R}$ is in some sense small, so that the series converges quickly and can be truncated. Define the nth order approximation as

$$\psi_n = \sum_1^n L^{-1}\left(RL^{-1}\right)^{n-1} \psi_{\rm inc}. \tag{12}$$

(Note that L and R depend on surface geometry and wavenumber only, not on incident field; and that one might expect convergence of the series (11) for given $\psi_{\rm inc}$ but not uniform (norm) convergence of the series (1).)This corresponds physically to an assumption that surface-surface interactions are dominated by those 'from the left', as expected in this scattering regime. L is large compared with R first, because L includes the dominant 'diagonal' value; second because a predominantly right-going wave gives rise to more rapid phase-variation in the integrand in R than in L. (Although this depends on surface geometry and cannot in general be quantified precisely, it occurs because in (5) the phase in Green's function kernel decreases as the observation point is approached from the left and then increases to the right; whereas the phase of ψ tends to increase throughout, like that of the incident field.) This is borne