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Generalized finite difference method for two-dimensional shallow water equations

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ABSTRACT

A novel meshless numerical scheme, based on the generalized finite difference method (GFDM), is proposed to accurately analyze the two–dimensional shallow water equations (SWEs). The SWEs are a hyperbolic system of first-order nonlinear partial differential equations and can be used to describe various problems in hydraulic and ocean engineering, so it is of great importance to develop an efficient and accurate numerical model to analyze the SWEs. According to split-coefficient matrix methods, the SWEs can be transformed to a characteristic form, which can easily present information of characteristic in the correct directions. The GFDM and the second-order Runge-Kutta method are adopted for spatial and temporal discretization of the characteristic form of the SWEs, respectively. The GFDM is one of the newly-developed domain-type meshless methods, so the time-consuming tasks of mesh generation and numerical quadrature can be truly avoided. To use the moving-least squares method of the GFDM, the spatial derivatives at every node can be expressed as linear combinations of nearby function values with different weighting coefficients. In order to properly cooperate with the split-coefficient matrix methods and the characteristic of the SWEs, a new way to determine the shape of star in the GFDM is proposed in this paper to capture the wave transmission. Numerical results and comparisons from several examples are provided to verify the merits of the proposed meshless scheme. Besides, the numerical results are compared with other solutions to validate the accuracy and the consistency of the proposed meshless numerical scheme.

1. Introduction

The flow fields of incompressible viscous fluid are governed by the well-known Navier-Stokes equations, which can be derived from the conservation law of mass and momentum. Under some assumptions [1], the shallow water equations can be derived by integrating the Navier-Stokes equations over the flow depth. In the past researches [2], the shallow water equations can be used to describe various applications in ocean and hydraulic engineering, such as the tidal oscillations in coastal and estuary water regions, open-channel flow in natural rivers and prismatic channels, flood waves generated by extreme storms or failure of dams, etc. The shallow water equations are a hyperbolic system of first-order nonlinear partial differential equations, so it is non-trivial to develop a numerical model to accurately and efficiently obtain solutions of the shallow water equations. In the past decades, many numerical schemes have been proposed to solve the shallow water equations, such as the finite difference method (FDM) [3,4], the finite volume method (FVM) [5,6], the finite element method (FEM) [7], the radial basis functions (RBFs) collocation method (RBFCM) [8,9], the local RBFs differential quadrature method (LRBFDQM) [10], etc. In this paper, we proposed a novel meshless scheme for numerical solutions of the shallow water equations without the need for time-consuming mesh generation and numerical quadrature.

Since the shallow water equations are a hyperbolic system of first-order nonlinear partial differential equations, the information of characteristic and the correct directions of wave transmission are quite important to numerical simulation. On the basis of the characteristic theory, Moretti [11] proposed an explicit FDM, which is called the λ-scheme, to accurately analyze the hyperbolic system of Euler equations of compressible flow. The wave transmission and their directions are properly present in the λ-scheme. After the successful combination of the FDM and the characteristic theory in the λ-scheme, Gabutti [12] improved the λ-scheme and numerically compared his proposed explicit finite difference scheme with other methods. The Gabutti scheme [12] is accomplished by diagonalizing and splitting the coefficient matrices of the non-conservation form of the shallow water equations. Therefore, the information of characteristic is described in

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each part of positive and negative eigenvalues of the coefficient matrices. Once the shallow water equations are transformed to the characteristic form, the predictor-corrector method, which includes three sequential steps, is adopted for the integration along time axis. Because the Gabutti scheme can accurately capture the information of wave transmission, Fennema and Hanif Chaudhry [3,13] adopted the Gabutti scheme to solve the Saint-Venant equation and the shallow water equations. Several examples of one-dimensional and two-dimensional hydraulic problems are present in their papers and the Gabutti scheme can accurately acquire satisfactory numerical results. Since both of the λ-scheme and the Gabutti scheme considered the characteristic theory in the coefficient matrices, these two schemes belonged to the category of split-coefficient matrix methods [2,3]. In the proposed meshless scheme, the concept of the split-coefficient matrix methods is adopted to transform the governing equations in order to include the information of wave transmission. Then, the generalized finite difference method (GFDM) and the second-order Runge-Kutta method are adopted for spatial and temporal discretizations, respectively.

With the rapid developments of computer software and hardware in the past half century, some mesh-based numerical methods for spatial discretization have been developed and applied to various engineering problems, such as the FDM, the FEM, the FVM, etc. In order to get rid of the time-consuming tasks of mesh generation and numerical quadrature in the mesh-based methods, many so-called meshless/meshfree methods are proposed, such as the method of fundamental solutions [14–16], the Trefftz method [17,18], the singular boundary method [19,20], the GFDM [21–28], the RBFCM [8,9], the local RBFCM [29,30], the LRBFDQM [10], etc. Among them, the GFDM is one of the most-promising domain-type meshless methods. While the GFDM is used to numerically analyze boundary-value problems, only two sets of randomly-distributed nodes are required. One is the set of boundary nodes and the other is the set of interior nodes. Once the spatial coordinates of theses nodes are obtained, the spatial derivatives at every node can be expressed as linear combinations of nearby function values with different weighting coefficients by using the moving-least squares method. To enforce the satisfactions of governing equations at every interior node and boundary conditions at every boundary node can yield the final sparse system of algebraic equations. The numerical solutions can be obtained by solving this system of algebraic equations. From the above-described numerical procedures of the GFDM for boundary value problems, it can be found that the GFDM is truly free from mesh generation and numerical quadrature. Besides, the concept of star in the GFDM can avoid the problems of ill-conditioning matrices, which usually appear in other meshless methods. So, it is obvious that the GFDM remains the advantages from both of the mesh-based methods and the meshless methods.

The explicit formulas of the GFDM are proposed by Benito et al. [21] in 2001 and then Gavete et al. [22] numerically investigated some factors of the GFDM. Since the GFDM is originated from the classical FDM and keeps the merits of mesh-based methods and meshless methods, it has been, in the past few years, applied to analyze various problems, such as parabolic and hyperbolic equations [23], the obstacle problems [24], the inverse problems [25,26], the density-driven groundwater flows [27], sloshing phenomenon [28], etc. From these GFDM-related researches, it is apparently revealed that the GFDM has great potential to be used for realistic engineering problems. Thus, in this paper, we proposed a meshless numerical scheme, based on the GFDM, to accurately and efficiently solve the shallow water equations. In addition, a new way to determine the shape of star in the GFDM is proposed in this paper in order to capture the correct transmission of characteristic.

The proposed meshless numerical scheme is the combination of the split-coefficient matrix method, the GFDM and the second-order Runge-Kutta method. At the beginning, the shallow water equations are transformed under the consideration of the split-coefficient matrix method. The coefficient matrices are decomposed according to the signs of eigenvalues, which denote the directions of wave transmission. Then, the GFDM and the second-order Runge-Kutta method are responsible for spatial and temporal discretization, respectively. Since the directions of transmission wave are marked in the coefficient matrices of partial differential equations, the conventional method to determine the shape of star in the GFDM is no longer suitable. For every node, a star is formed by choosing the nearest nodes, so the shape of star in the GFDM conceptually resembles to a circular disk for a two-dimensional problem and the considered node is usually located at the center of the star. In order to include the direction of characteristic, we proposed a new way to determine the shape of star. In the proposed method, if the wave is transmitted from left-hand side, the shape of star is only the left half disk. One the other hand, if the wave is transmitted from right-hand side, the shape of star is only the right half disk. The similar procedures can be extended to waves from the upward and downward directions. Although it might take more computational cost to calculate the weighting coefficients due to different shapes of star in the beginning of simulation, the information of wave transmission can be accurately analyzed during the entire simulation. To the best of the authors' knowledge, it is the first time that the shape of star at every node in the GFDM is modified under the consideration of transmission of characteristic.

After the introduction of motivation of this study and the discussions of relevant researches, the governing equations and the proposed numerical procedures are described in the next sections. In the section of numerical results and comparisons, several examples are provided to verify the merits of the proposed meshless numerical method and the numerical results are compared with other solutions. Finally, some conclusions and discussions will be drawn according to the results and comparisons.

2. Shallow Water Equations

In the realm of hydraulic, ocean and atmospheric engineering, the shallow water equations can be used to explain various engineering applications. The shallow water equations can be derived from the Navier-Stokes equations by using vertically-averaged quantities. The non-conservative form of the shallow water equations are depicted as follows [2,3]:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{P}\frac{\partial \mathbf{V}}{\partial x} + \mathbf{H}\frac{\partial \mathbf{V}}{\partial y} + \mathbf{T} = 0, \tag{1}$$

in which

$$\mathbf{V} = \begin{bmatrix} h \\ u \\ v \end{bmatrix},\ \mathbf{P} = \begin{bmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{bmatrix},\ \mathbf{H} = \begin{bmatrix} v & 0 & h \\ 0 & v & 0 \\ g & 0 & v \end{bmatrix},\ \mathbf{T} = \begin{bmatrix} 0 \\ gS_{0x} - gS_{fx} \\ gS_{0y} - gS_{fy} \end{bmatrix}. \tag{2}$$

In Eqs. (1) and (2), h denotes the water depth. u and v are the vertically-averaged velocity components in x and y directions. g is the acceleration due to gravity. S_{0x} and S_{0y} are channel bottom slope in the x and y directions. S_{fx} and S_{fy} are the slopes of the energy grade lines in the x and y directions and can be demonstrated as:

$$S_{fx} = \frac{n^2 u\sqrt{u^2 + v^2}}{h^{4/3}}, \tag{3}$$

and

$$S_{fy} = \frac{n^2 v\sqrt{u^2 + v^2}}{h^{4/3}}, \tag{4}$$

where n is the Manning's roughness coefficient. $H = h + z$ denotes the water level and z is the bed elevation.

The matrices **P** and **H** in Eq. (1) have very important property, related to the information of characteristic, and their eigenvalues are given by