Engineering Analysis with Boundary Elements 80 (2017) 184–198

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A family of exponentially-gradient elements for numerical computation of singular boundary value problems

Morteza Eskandari-Ghadi[a,*], Delaram Mehdizadeh[a], Ali Morshedifard[b], Mohammad Rahimian[a]

[a] School of Civil Engineering, College of Engineering, University of Tehran, P.O Box 11155-4563, Tehran, Iran
[b] Department of Civil and Environmental Engineering, University of California at Irvine, Irvine, CA 92617, USA

ARTICLE INFO

Keywords:
Exponentially gradient elements
Singularity
High gradient
Mixed boundary value problem
Boundary element method
Finite element method

ABSTRACT

In this paper, a new family of single-parameter exponentially gradient elements (EG-elements) are introduced, which can be used in various numerical procedures such as boundary and finite element methods. These elements have the ability to accurately interpolate the unknown values in regions, where either high gradient or singularity of the unknown field occurs. The shape functions of two-dimensional EG-elements are high gradient at either a corner of the element or at an edge of the element. Another advantage of this element is that the regular quadratic shape functions are obtained as a special case of EG-elements by adjusting the single parameter of the element, which allows this element to be used as regular Lagrange quadratic element, where it is appropriate. Some mixed boundary value problems are solved with the use of EG-elements in a boundary element program to show the capability of these elements for capturing the solution with less number of elements and higher accuracy.

1. Introduction

Many kinds of Mixed Boundary Value Problems (MBVP) are encountered in engineering mechanics [1–5] and potential theory [6,7], where the natural boundary conditions are discontinuous and thus the derivatives of the solution for the MBVP is singular at discontinuities. Contact problems, crack investigation [7], scattering problems and problems in fracture mechanics are some examples in mechanics, and electrostatic potential of plates charged to a prescribed potential or electrostatic field due to two parallel plates are examples in potential theory [6]. When an analytical solution is sought for this type of MBVP, a very precise attention should be paid, and when it is considered to be solved numerically, some difficulties are encountered both in reproducing the singularities of the problem and in numerical integrations [7,8]. Because of these issues, mathematicians and engineers are interested in this kind of problems. In addition, the numerical research group is also interested in the phenomenon. There exist a few researches for either numerical integration of singular functions or numerical solutions for singular integral equations [7–11], many of which proposed to express the unknown function in terms of linear combination of some known functions, such as Chebyshev [9] and Jacobi [12,13] series functions. When one expresses an unknown function, say f, in the form of linear combination of some other known functions in its entire domain, where both regular and singular behavior of f are seen, it should be noticed that a large number of the known functions should be used to capture all the behavior of f. On the other hand, if f is expressed in terms of some function in a subset of its domain, then a small number of the known functions may be adequate. Polynomials from constant to higher degree such as 3rd or 4th degree are used in ordinary finite element method, called Lagrange elements. Since, these elements due to their shape functions are naturally smooth, they cannot capture the singular or high gradient behavior present in the solution of various MBVPs appropriately and a large number of elements are needed to have a poor approximation for the solution. Pak and Ashlock [14] introduced a family of two-parameter power based adaptive elements which can be adjusted to capture both regular and high-gradient variations of different functions. Although the shape functions they derived can be used for numerical analysis of some high-gradient functions, however, a combination of two real parameters should be selected for adjusting the behavior of the element. On the other hand, the complicated form of the kernel function used in deriving the shape functions, makes them less attractive for applications.

In this work, we are going to introduce a new one-parameter family of shape functions with an adaptive behavior to capture both regular and singular behavior of different phenomena. Since, integration of singular functions needs some special attention, nonsingular, however high gradient shape functions are used in this new element. To this

* Corresponding author.
E-mail address: ghadi@ut.ac.ir (M. Eskandari-Ghadi).

http://dx.doi.org/10.1016/j.enganabound.2017.03.013
Received 4 November 2016; Received in revised form 20 January 2017; Accepted 9 March 2017

end, an exponential function is used as the kernel function of the new element. Since, both the exponential function and its derivatives are linearly dependent, some extra terms are added to the shape functions to make a more complete basis, in the sense of vector space [see [15,16]], for the new element. On the other hand, since the main part of the gradient of the new elements is expressed by the exponential function, we call them the Exponentially Gradient-elements (EG-elements). The EG-elements in this paper are introduced in such a way that their behavior from regular to very high gradient can be controlled with only one parameter, and the so called Lagrange element can be derived as a special case of the EG family of elements. In this way, any EG-element introduced in this paper will be an adaptive element. Some two- and three-dimensional contact boundary value problems are numerically solved with the application of EG family of elements in a boundary element method to show the power of EG-family line and surface elements.

2. Basis for construction of EG-family elements

A linear vector (function) space is a set containing some vectors (functions) that except some other properties satisfy two main axioms, which are closure under addition and closure under multiplication by real numbers [see [15,16]]. A Finite Dimensional Linear Vector Space (FDLVS) may be a subset of a linear vector (function) space that is spanned on a basis with finite members. An arbitrary vector, say $\boldsymbol{u}$, can be exactly expressed in an FDLVS if there exists a basis for the FDLVS containing $\boldsymbol{u}$, otherwise, $\boldsymbol{u}$ cannot be exactly expressed as a linear combination of the members of the FDLVS, and thus any linear combination of the members of FDLVS is an approximation for $\boldsymbol{u}$. Any smooth function can be expressed in an FDLVS that spanned on a basis containing smooth functions, however, a high gradient or singular function cannot be well expressed on the domain of interest in an FDLVS that is spanned on a basis containing only smooth functions with low gradient behavior. In this case, high gradient functions are needed to make a good approximation for the singular functions. There are many cases in computational mechanics that high gradient or singular functions are encountered. Thus, an FDLVS spanned on a basis containing high gradient functions is needed to handle the singular functions. On the other hand, the functions in the basis of an FDLVS should be continuous on the domain of interest to be a place for making good approximations for continuous functions. In this paper, we are going to present an FDLVS spanned on a basis containing only three functions with an arbitrary gradient, where the gradient is adjusted with only one parameter. This FDLVS is used for making a new three-node finite element used in one-dimensional boundary value problems. With the use of standard methods, the two-dimensional high gradient elements are made. The Lagrange three-node one-dimensional element will be a special case of the proposed element.

3. One-dimensional EG-elements

A family of single parameter Exponentially-Gradient (EG) shape functions is introduced, which can be utilized for making one- to three-dimensional finite and boundary elements. The shape functions are adjusted in such a way that they can vary from regular to very high gradient behavior. To do so, first the following interpolation function is presented for derivation of the shape functions of a three-node one-dimensional element:

$$f(s;\ m) = A + Bs + C\frac{s(s+1)}{2}\mathrm{e}^{m(s-1)} \tag{1}$$

where s is the parent coordinate in the domain $[-1,\ 1]$ (see Fig. 1), A, B and C are constants, which will be derived from the nodal values of the variable being interpolated and m is the sole EG-parameter that can be chosen in such a way that a desired behavior is obtained from the

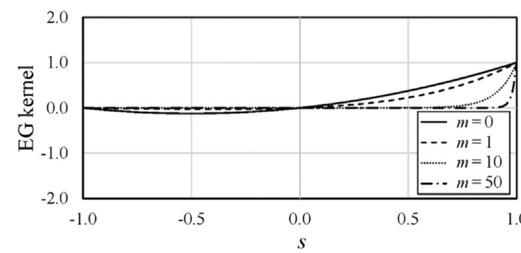

Fig. 1. The EG-kernel for different values of m.

function $f(s;\ m)$. We call the term $\frac{s(s+1)}{2}\mathrm{e}^{m(s-1)}$ in the interpolation function $f(s;\ m)$ as the EG-kernel function, which can be localized and sharpened toward the EG end of the parent domain via the m parameter, as can be seen from Fig. 1. As is clear from the formulation of the EG-kernel function and as seen from Fig. 1, the EG-kernel function is smooth everywhere for different values of m, however, the larger the value of m the higher the gradient of the kernel function is achieved.

If the usual nodal requirements for an element with three nodes are imposed, the following equations are obtained:

$$\begin{aligned} f(-1) &= A - B = f_1 \\ f(0) &= A = f_2 \\ f(1) &= A + B + C = f_3 \end{aligned} \tag{2}$$

where f_1, f_2 and f_3 are the nodal values of the variable being interpolated. A simultaneous solution of these three equations gives the constants A, B and C in terms of f_1, f_2 and f_3 as follows:

$$A = f_2, \quad B = f_2 - f_1, \quad C = f_1 - 2f_2 + f_3 \tag{3}$$

Substitution of Eq. (3) into Eq. (1) makes it possible to rearrange the interpolated function in the form of $f(s;\ m) = N_1(s;\ m)f_1 + N_2(s;\ m)f_2 + N_3(s;\ m)f_3$ where $N_i(s)$, $(i = 1, 2, 3)$ are the EG-shape functions for a 3-node one-dimensional EG-element obtained as

$$\begin{aligned} N_1^{EG} &= \frac{1}{2}s(-2 + \mathrm{e}^{m(s-1)}(s+1)) \\ N_2^{EG} &= 1 + s - \mathrm{e}^{m(s-1)}s(s+1) \\ N_3^{EG} &= \frac{1}{2}\mathrm{e}^{m(s-1)}s(s+1) \end{aligned} \tag{4}$$

The EG-shape functions should be selected in such a way that they can make a basis for an FDLVS containing completely regular to very high gradient functions defined on a finite subset of real numbers. In this way, one can use linear combinations of those shape functions to describe any regular to very high gradient functions defined on the subset. Of course, the shape functions should possess the delta function property, the partition of unity property, C°-continuity, and consistency [17] conditions. The delta function property implies that the shape functions should have unit values at their home node and vanish at remote nodes of the element, while the partition of unity property ensures that the shape functions should sum to unity. The C°-continuity at the subset makes sure of the continuity of any function directly expressed by the shape functions, and consistency condition requires that the interpolation function represents exactly any polynomial function up to some order. Because of analyticity of the exponential function, the shape functions used in the EG-elements are smooth with any value of m, however, their gradient is controlled by the value of m. Examples of the shape functions are shown in Fig. 2 for a wide range of the parameter m. As it can be seen from the display, they exhibit exactly the kind of localizability and adaptability that are lacking in current shape function ensembles. An important property that can be observed is that the shape functions of the one-dimensional Lagrange quadratic element can be easily reproduced by choosing