Engineering Analysis with Boundary Elements 74 (2017) 14–18

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A regularization scheme applied to the direct interpolation boundary element technique with radial basis functions for solving eigenvalue problem

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ARTICLE INFO

Keywords:
Helmholtz problems
Eigenvalue
Boundary element method
Interpolation
Radial basis functions

ABSTRACT

This paper shows a regularization scheme applied to the recently developed Direct Interpolation Technique with Radial Basis Functions (DIBEM) for elimination of the singularity that exists in the kernel of the domain integral. As a simple interpolation, the kernel is approximated directly in DIBEM; however, it is composed of the fundamental solution, distinct positions between the source points and the field points being thus required. Through the proposed regularization scheme, both sets of source points and field points, as well as base points used for interpolation with radial functions may have the same coordinates. This facilitates the data entry and also the implementation of several operational steps of the DIBEM formulation. Solution of eigenvalue problem, generated by the Helmholtz Equation, is here chosen to exemplify the efficacy of the regularization procedure, but many other problems can thus be addressed, particularly the diffusive-advective problem, that has higher level of singularity in the interpolated kernel.

1. Introduction

The direct interpolation boundary element method (DIBEM) is a new technique, suitable for solving partial differential equations whose operators are not self-adjoint [1] or then cases in which the deduction of the associated fundamental solution [2] is very difficult.

It is an alternative to the well known Dual Reciprocity Technique (DRBEM) [3]. Some examples of application are cases which involve sources, domain actions, diffusive-advective problems, inertia forces and other cases.

The DIBEM procedure has been successfully applied to Poisson problems [4] and Helmholtz problems [5]. This technique applies an approximation procedure using radial basis functions [6,7] to domain integrals comprised to non self-adjoint kernels, that is relatively similar to that of DRBEM; however, it is yet simpler, more general and more robust, since the formulation proposed here does not require construction of two auxiliary matrices by multiplying classic boundary element matrices H and G.

Unlike the DRBEM, the entire kernel of the domain integral is interpolated, including the fundamental solution. Thus, the coincidence in positions of source points $\boldsymbol{\xi}$ and field points $\mathbf{X}$ produces singularity. To avoid this, preliminary DIBEM formulation establishes that the coordinates of source points and field points must be taken distinctly, what means two sets of data coordinates entry.

The proposal here is to use an idea based on the concept of Hadamard [8] to eliminate the singularities, making both the DIBEM implementation process and data entry faster, since field points and source points can coincide. Obviously, it is necessary to check the quality of the results, since the original procedure, performed well.

2. Basic formulation

The Helmholtz Equation can be interpreted as a simplification of the Acoustic Wave Equation [9] given by:

$$\nabla^2 U(\mathbf{X}, t) = \frac{1}{k^2}\ddot{U}(\mathbf{X}, t) \tag{1}$$

In Eq. (1) k is the velocity of propagation of the acoustic wave and

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http://dx.doi.org/10.1016/j.enganabound.2016.10.008
Received 14 March 2016; Received in revised form 2 August 2016; Accepted 11 October 2016
Available online 02 November 2016

U(**X**,t) is the spatial response of the system to any excitations with generalized modal content. Thus, in the particular case in which one seeks the response produced in the system by a variable excitation whose frequency ω is known, the potential U(**X**,t) is a response composed for specific vibrational content, in a following form:

$$U(\mathbf{X}, t) = u(\mathbf{X})e^{-i\omega t} \tag{2}$$

In Eq. (2),"i" is the imaginary unit and u(**X**) is the stationary amplitude spatial response of the system to the harmonic excitation of frequency ω. Substitution of Eq. (2) in Eq. (1) results in a model that the potential u(**X**) varies from point-to-point on the domain field, comprising the Helmholtz Equation, that is:

$$\nabla^2 u(\mathbf{X}) = \frac{-\omega^2}{k^2} u(\mathbf{X}) \tag{3}$$

Integrating Eq. (3) over the physical domain Ω(X), using an auxiliary fundamental solution u*(ξ;**X**), as shown below, the following equation results:

$$\int_\Omega \nabla^2 u(\mathbf{X}) u^*(\xi; \mathbf{X}) d\Omega = \frac{-\omega^2}{k^2} \int_\Omega u(\mathbf{X}) u^*(\xi; \mathbf{X}) d\Omega \tag{4}$$

In the DIBEM procedure, the fundamental solution does not depend of the frequency, since it is related to the Laplace's problem. This simplification allows the DIBEM to generate an explicit inertia matrix, as occurs with the Finite Element Method [10] and the DRBEM [11], at the expense of a loss of representation capacity of the functional space generated in the BEM integral formulation.

Thus, the basic BEM approach to the Laplacian operator is maintained and the application of integration by parts and the Divergence Theorem on Eq. (4), operations well documented in literature [12], gives the following integral expression:

$$c(\boldsymbol{\xi})u(\boldsymbol{\xi}) + \int_\Gamma u(\mathbf{X}) q^*(\boldsymbol{\xi}; \mathbf{X}) d\Gamma - \int_\Gamma q(\mathbf{X}) u^*(\boldsymbol{\xi}; \mathbf{X}) d\Gamma = \frac{\omega^2}{k^2} \int_\Omega u(\mathbf{X}) u^*(\boldsymbol{\xi}; \mathbf{X}) d\Omega \tag{5}$$

In Eq. (1), u(**X**) is the scalar potential and q(**X**) is its normal derivative; u*(**ξ**,**X**) is the fundamental solution for the Laplace's equation and q*(**ξ**,**X**) is its normal derivative;. The coefficient c(**ξ**) depends on the position of point **ξ** in the physical domain Ω(**X**)+Γ(**X**), and if the point is located on the boundary Γ(**X**), it also depends on its smoothness [13].

The DIBEM can be used in other relevant problems that belong to the Generalized Scalar Field Theory [14]. The typical BEM approach to the Laplacian operator is maintained, i.e., the fundamental solution of stationary diffusive problem is used as an auxiliary function.

Using DIBEM, the complete kernel of the domain integral is directly interpolated using radial basis functions according to Eq. (6):

$$u(\mathbf{X})u^*(\boldsymbol{\xi}; \mathbf{X}) \cong {}^{\xi}\alpha_i F^i(\mathbf{X}^i; \mathbf{X}) \tag{6}$$

Because the interpolation functions $F^i(\mathbf{X}^i, \mathbf{X})$ belong to the class of radial functions, their argument are composed of the Euclidean distance $r(\mathbf{X}^i, \mathbf{X})$, which characterises the positions of the base points $\mathbf{X}^i$ relative to generic domain points **X**.

It should be highlighting that the domain integral on the right-hand-side of Eq. (5) is regular, since the integral of a logarithm function is an improper convergent integral. However, a logarithm function tends to infinity when its argument tends to zero. This is a fundamental aspect regarding the DIBEM, since the kernel is comprised by fundamental solution and the approximation by linear combination of radial basis functions is done prior to performing the domain integration. So, it is necessary to avoid position coincidence between field points and base points.

After the discretisation process, these points **X** are used to generate the node points, at which the u(**X**) potential is calculated. For each source point **ξ**, the interpolation given by Eq. (6) corresponds to scanning all points $\mathbf{X}^i$ in relation to domain points **X**, weighted by the coefficients ${}^{\xi}\alpha^i$. Hence, the ${}^{\xi}\alpha^i$ coefficients can be obtained by solving a system of algebraic equations, as shown ahead.

Similarly to DRBEM, the proposed method also uses a primitive interpolation function $\Psi^i(\mathbf{X}^i, \mathbf{X})$; thus, the integral of the source term expressed by the right-hand-side of Eq. (5) becomes the following:

$$\int_\Omega {}^{\xi}\alpha^i F_i(\mathbf{X}^i; \mathbf{X}) d\Omega = \int_\Omega {}^{\xi}\alpha_i \psi^i_{,jj}(\mathbf{X}^i; \mathbf{X}) d\Omega = \int_\Gamma {}^{\xi}\alpha_i \psi^i_{,j}(\mathbf{X}^i; \mathbf{X}) n_j(\mathbf{X}) d\Gamma = {}^{\xi}\alpha_i \int_\Gamma \eta^i(\mathbf{X}^i; \mathbf{X}) d\Gamma \tag{7}$$

The boundary transformation represented in Eq. (7) was tested yielding satisfactory results in preliminary applications that consisted of calculations of volumes and image of functions using several kind of functions, two of them being paraboloid and Franke [15]. Also simulations were performed with the BEM to solve Poisson problems; some of which included interpolation based on radial basis functions with compact support [16].

3. Regularization procedure

The regularization scheme avoids elegantly the coincidence between field and basis points, without resorting to a computational algorithm that simply prevents the coincidence of these points, since it would result in impaired quality of the interpolation.

Hadamard regularization concept is usually applied in a different context [17], nevertheless its idea can be used here to exclude the singularity in Eq. (6) when the positions of source and field points are coincident. Thus, the following procedure is applied to the right-hand-side of Eq. (5):

$$c(\boldsymbol{\xi})u(\boldsymbol{\xi}) + \int_\Gamma u(\mathbf{X}) q^*(\boldsymbol{\xi}; \mathbf{X}) d\Gamma - \int_\Gamma q(\mathbf{X}) u^*(\boldsymbol{\xi}; \mathbf{X}) d\Gamma = \frac{1}{k^2}\omega^2 \{ \int_\Omega [u(\mathbf{X}) u^*(\boldsymbol{\xi}; \mathbf{X})] d\Omega - \int_\Omega [u(\boldsymbol{\xi}) u^*(\boldsymbol{\xi}; \mathbf{X})] d\Omega \} + \frac{1}{k^2}\omega^2 \int_\Omega [u(\boldsymbol{\xi}) u^*(\boldsymbol{\xi}; \mathbf{X})] d\Omega \tag{8}$$

The two first integrals on the right-hand-side of Eq. (8) are approximated together as follows:

$$\frac{1}{k^2}\omega^2 \{ \int_\Omega [u(\mathbf{X}) u^*(\boldsymbol{\xi}; X)] d\Omega - \int_\Omega [u(\boldsymbol{\xi}) u^*(\boldsymbol{\xi}; X)] d\Omega \} \approx \frac{1}{k^2}\omega^2 \{ \int_\Omega {}^{\xi}\alpha^i F_i(\mathbf{X}^i; \mathbf{X}) d\Omega \} \tag{9}$$

Using the DIBEM transformation, given by Eq. (7), one has:

$$c(\boldsymbol{\xi})u(\boldsymbol{\xi}) + \int_\Gamma u(\mathbf{X}) q^*(\boldsymbol{\xi}; X) d\Gamma - \int_\Gamma q(\mathbf{X}) u^*(\boldsymbol{\xi}; X) d\Gamma = \frac{1}{k^2}\omega^2 \{ {}^{\xi}\alpha_i \int_\Gamma \eta^i(\mathbf{X}^i; \mathbf{X}) d\Gamma \} + \frac{1}{k^2}\omega^2 \int_\Omega [u(\boldsymbol{\xi}) u^*(\boldsymbol{\xi}; X)] d\Omega \tag{10}$$

For convenience, temporarily the second term on the right-hand-side of Eq. (10) should be ignored. The mathematical treatment and discretisation of the remaining terms are given in detail in a previous paper [5], where the following matrix system is achieved:

$$\begin{pmatrix} H_{11} \cdots H_{1n} \\ \cdots \\ H_{n1} \cdots H_{nm} \end{pmatrix} \begin{pmatrix} u_1 \\ u_n \end{pmatrix} - \begin{pmatrix} G_{11} \cdots G_{1n} \\ \cdots \\ G_{n1} \cdots G_{nm} \end{pmatrix} \begin{pmatrix} q_1 \\ q_n \end{pmatrix} = \frac{\omega^2}{k^2} \begin{pmatrix} {}^{1}\alpha_1 \cdots {}^{1}\alpha^m \\ \cdots \cdots \cdots \\ {}^{n}\alpha_1 \cdots {}^{n}\alpha_m \end{pmatrix} \begin{pmatrix} N^1 \\ \cdots \\ N^m \end{pmatrix} = \frac{\omega^2}{k^2} \begin{pmatrix} A^1 \\ \cdots \\ A^n \end{pmatrix} \tag{11}$$

Each coefficient A_ξ is given explicitly by:

$$A_\xi = (N_1 \quad N_2 \ldots \quad N_m) \begin{pmatrix} {}^{\xi}\alpha_1 \\ \cdots \\ {}^{\xi}\alpha_m \end{pmatrix} \tag{12}$$

In turn, the ${}^{\xi}\boldsymbol{\alpha}$ can be calculated using the basic interpolation equation, that is:

$$[{}^{\xi}\boldsymbol{\alpha}] = [F]^{-1}[{}^{\xi}\boldsymbol{\Lambda}][F]\boldsymbol{\alpha} = [F]^{-1}[{}^{\xi}\boldsymbol{\Lambda}][u] \tag{13}$$

Thus, in this case: