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Global error analysis of two-dimensional panel methods for dirichlet formulation

José M. Ezquerro, Ana Laverón-Simavilla, Victoria Lapuerta*, Jeff Porter

Universidad Politécnica de Madrid, Plaza de Cardenal Cisneros, 3, 28040 Madrid, Spain

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ABSTRACT

A rigorous analytical study of the global error of panel methods is presented. The analysis is performed for a wide variety of body shapes and different panel geometries to fully understand their effect on the convergence of the method. In particular, we study the global error associated with panel methods applied to thin or thick bodies with purely convex parts or with both convex and concave parts, and with smooth or non-smooth boundaries. Most previous studies focused on the analysis of local error, considering only the influence of the nearest panels and excluding the rest. The difference is shown to be appreciable in many configurations. Generally, there is a lack of consensus concerning the order of magnitude of the error for panel methods even in the simplest case with flat panels and a constant distribution of doublets along them. This paper clarifies apparently different or inconsistent results obtained by other authors.

1. Introduction

The importance of the Laplace equation in aerodynamics (and many other fields of science) means that a great deal of effort has been directed toward developing analytical and numerical methods for its solution. Among the most popular numerical schemes are panel methods, or boundary element methods (BEMs) [1,2], which reduce the problem of finding the velocity potential for the entire fluid to the calculation of this potential on the surface of the body itself. Thus, the dimension of the problem is reduced from three to two (or, in the case of two-dimensional flows, from two to one) making BEMs very attractive for their low computational cost. Since the pioneering work of Hess and Smith there have been numerous publications and many numerical codes based on panel methods [3–14]; among these we emphasize the reviews of Hess [9], Erickson [10] and the book of Katz and Plotkin [12]. Boundary element methods are an active field of study, especially within the engineering community, with new applications being developed rapidly.

The panel method based on Green's formula was first introduced in the work of Morino and Kuo [4], in which the primary unknown was the velocity potential. There are two main formulations both based on Green's formula: Neumann and Dirichlet [12]. The Dirichlet formulation solves the Laplace equation numerically and provides the velocity potential. However, with the Neumann formulation, only differences of potential are obtained. The Dirichlet formulation is more stable and more suitable to numerical computation than the Neumann formulation and leads to numerical errors of a smaller order of magnitude.

Initially, panel methods were developed using flat panels and a constant [3,5] or linear [6] distribution of singularities on each panel. Beginning in the 1970's however, singularity distributions were also modeled (on each panel) using quadratic [4,15,16] or cubic [17] functions. In a similar fashion, the panels themselves, which were initially taken to be flat, were generalized to include non-planar geometries [4,15]. However, in the last several decades many numerical codes returned to the original low-order approach, as indicated in [12]. The main reason for this is the more complicated implementation of higher-order methods compared to lower-order ones [11].

A lot of studies have addressed the question of error in these methods. Some carry out a numerical analysis of the error by comparing a numerical solution with a known analytical solution [18–20]. Others perform a local analysis of the error by using small curvature expansions to obtain local approximations to the velocity and potential integrals [21–24]. However, to the best of our knowledge, a rigorous analytical study of the global error of these methods that applies to thin or thick bodies, with purely convex parts or with both convex and concave parts, and with smooth or non-smooth boundaries, has not yet been performed. In this paper, we present such an analysis for a wide variety of body shapes and try to understand the effect of the body and panel geometry on the convergence of the method. This allows us to clarify several important questions about the convergence

* Corresponding author.
E-mail addresses: jezquerro@eusoc.upm.es (J.M. Ezquerro), ana.laveron@upm.es (A. Laverón-Simavilla), mariavictoria.lapuerta@upm.es (V. Lapuerta), jeff.porter@upm.es (J. Porter).

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rate for the velocity potential since, even for the simplest case of flat panels with a constant distribution of doublets along them, there is a lack of general consensus. Depending on the airfoil geometry, the panel geometry and the discretization, apparently different or inconsistent results are obtained by other authors, and differences between theoretical and numerical results exist as well [18–20,24,25].

This work presents a formal analytical and numerical analysis of the asymptotic global error in panel methods when applied to a Dirichlet formulation [12] for different body geometries. In addition, an analysis of the influence of the panel geometry on the global error is performed. The work is organized as follows. In Section 2 a brief description of panel methods is given. In Section 3 the global error analysis is performed analytically. In Section 4 the details of the error estimation are presented. Section 5 considers the numerical and analytical solutions for different body geometries in order to compare the actual and predicted errors in each case. Finally, in Section 6 the main conclusions are given.

2. Brief description of the panel method

The velocity potential around a body of known shape submerged in a potential flow satisfies the irrotational, incompressible continuity equation in the body's frame of reference [12]:

$$\nabla^2 \Phi = 0. \tag{1}$$

Boundary conditions require a vanishing normal velocity component on the body surface,

$$\nabla\Phi\cdot\boldsymbol{n} = 0, \tag{2}$$

and a constant velocity in the far field limit:

$$\lim_{r\to\infty} \nabla\Phi = \boldsymbol{V}_\infty, \tag{3}$$

where $\boldsymbol{V}_\infty$ denotes the imposed velocity far from the body.

Using Green's identity, the general solution to Eq. (1) can be written as:

$$\Phi(\boldsymbol{p}) = \int_{S_B}\left(\frac{\partial\Phi}{\partial n} - \frac{\partial\Phi_{\text{int}}}{\partial n}\right)\Phi_m\, ds - \int_{S_B}(\Phi - \Phi_{\text{int}})\nabla\Phi_m\cdot\boldsymbol{n}\, ds - \int_{S_W}(\Phi^+ - \Phi^-)\nabla\Phi_m\cdot\boldsymbol{n}\, ds + \Phi_\infty, \tag{4}$$

which gives the velocity potential Φ at any point $\boldsymbol{p}$. This potential is considered to be caused by a distribution, on the surface of the body S_B, of point sources of intensity $\partial\Phi/\partial n - \partial\Phi_{\text{int}}/\partial n$ and doublets of intensity $\Phi - \Phi_{\text{int}}$ oriented along axes $\boldsymbol{n}$, and by a distribution, along the wake, of doublets of intensity $\Phi^+ - \Phi^-$ with axis of orientation $\boldsymbol{n}$. Fig. 1 shows the body and the relevant surfaces; on the body $\boldsymbol{n}$ is oriented outward while it points upward along the wake. Φ_m is the velocity potential produced at a point $\boldsymbol{p}$ by a point source of unit strength located on ds, $\nabla\Phi_m\cdot\boldsymbol{n}$ gives the velocity potential at a point $\boldsymbol{p}$ produced by a doublet of unit strength located on ds and oriented along $-\boldsymbol{n}$, Φ_{int} is the so-called interior potential, which is required to satisfy the Laplace equation in the interior of the body, and the final term in Eq. (4) is the potential of the stationary flow far from the body, evaluated at $\boldsymbol{p}$: $\Phi_\infty = U_\infty(x\cos\alpha + z\sin\alpha)$, where $U_\infty = |\boldsymbol{V}_\infty|$, α is the angle between the incident flow and a reference line (angle of attack), and x and z are the coordinates of the point $\boldsymbol{p}$ in a fixed reference frame.

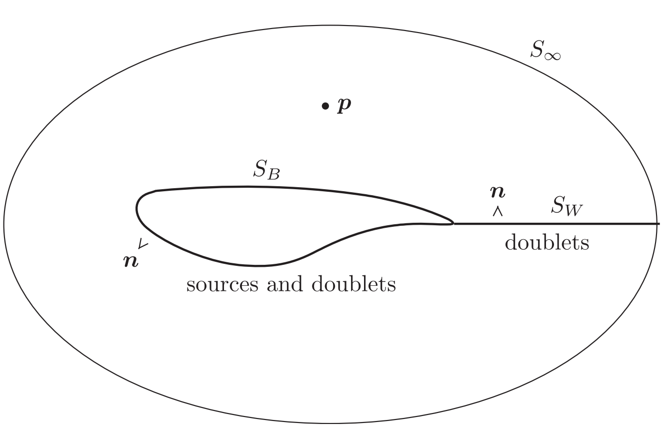

Fig. 1. Sketch of the body and associated surfaces: body surface S_B, wake surface S_W, and a surface at infinity S_∞.

Imposing $\partial\Phi/\partial n = 0$ on the boundary of the body, the velocity potential at the point $\boldsymbol{p}$ can be written as

$$\Phi(\boldsymbol{p}) = \int_{S_B}\sigma\Phi_m\, ds - \int_{S_B}\mu\nabla\Phi_m\cdot\boldsymbol{n}\, ds - \int_{S_W}(\Phi^+ - \Phi^-)\nabla\Phi_m\cdot\boldsymbol{n}\, ds + \Phi_\infty, \tag{5}$$

where

$$\sigma = -\frac{\partial\Phi_{\text{int}}}{\partial n}, \quad \mu = \Phi - \Phi_{\text{int}}. \tag{6}$$

Taking the point $\boldsymbol{p}$ to be on the surface of the body reduces the problem to an integral equation for the unknown velocity potential on the surface.

To simplify calculations, in what follows we take $\Phi_{\text{int}} = 0$. In this case the point source distribution σ vanishes, and Eq. (5) reduces to

$$\Phi(\boldsymbol{p}) = -\int_{S_B}\Phi\nabla\Phi_m\cdot\boldsymbol{n}\, ds - \int_{S_W}(\Phi^+ - \Phi^-)\nabla\Phi_m\cdot\boldsymbol{n}\, ds + \Phi_\infty. \tag{7}$$

3. Global error estimate for dirichlet formulation

Here we derive an estimate for the expected numerical error upon solving Eq. (7) with the lower order panel method. This equation can be written as

$$\Phi(\boldsymbol{p}) = \frac{1}{2\pi}\int_{S_B}\frac{\Phi(s)(\boldsymbol{p}-\boldsymbol{s})\cdot\boldsymbol{n}}{|\boldsymbol{p}-\boldsymbol{s}|^2}ds + \frac{\Gamma}{2\pi}\int_{S_W}\frac{(\boldsymbol{p}-\boldsymbol{\xi}_w)\cdot\boldsymbol{n}_w}{|\boldsymbol{p}-\boldsymbol{\xi}_w|^2}d\xi_w + \Phi_\infty(\boldsymbol{p}), \tag{8}$$

where, in the first integral, the variable of integration, s, is the arc length parameter along the body surface, $\boldsymbol{s} = \boldsymbol{s}(s)$ is a point on the body surface S_B, and $\boldsymbol{n} = \boldsymbol{n}(s)$ is the (unit) normal vector directed outward from that point. In the second integral the variable of integration is ξ_w, measuring distance along the wake panel S_W, while $\boldsymbol{\xi}_w = \boldsymbol{\xi}_w(\xi_w)$ is a point on the wake panel and $\boldsymbol{n}_w$ is a unit normal vector directed upwards. The prefactor $\Gamma = \Phi^+ - \Phi^-$ denotes the circulation around the body.

In what follows, the geometry of the body will be approximated with a collection of flat panels ℓ_i, $i = 1..N$ of length l_i. We assume that the intensity of the doublet distribution is constant on each individual panel and that all panels are of comparable size, i.e, with a characteristic lengthscale $l = O(1/N)$; hereafter, we use the Landau notation "$O(\cdot)$" for order of magnitude. The discretization of the body surface and the wake are illustrated in Fig. 2.

Enumeration of the panels begins at the point of attachment of the wake, with panel number 1, and continues clockwise around the body, ultimately reaching the starting point again after panel N (this time from above the wake panel). As illustrated in Fig. 2, the endpoints of these panels (which lie on the body surface) similarly divide the true body surface into N (curved) segments L_i. We may thus decompose the first integral term in Eq. (8) to get

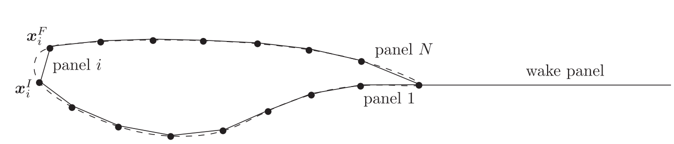

Fig. 2. Discretization of the body surface.