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Boundary-based finite element method for two-dimensional anisotropic elastic solids with multiple holes and cracks

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ABSTRACT

A special boundary element for the two-dimensional anisotropic elastic solids containing a single elliptical hole or crack is applied. The main feature of this special boundary element is that no meshes are needed along the hole or crack boundary. Take this special boundary element as a base, in this paper a new method called boundary-based finite element method is developed to deal with the problems of two-dimensional anisotropic elastic solids containing multiple holes and cracks. This method is established by using the relation between nodal force of finite element and surface traction of boundary element. With the aid of this relation, a combination of boundary elements can be transformed into a single finite element. By purposely arranging each subregion with a single hole or crack and assembling the entire region according to the rule of finite element method, the problems with multiple holes and cracks can be solved. Furthermore, simple formulae to evaluate the stress concentration factor of hole and the stress intensity factors of crack are derived, by which these factors can be evaluated by using only the remote boundary displacements and tractions. Accuracy and efficiency are illustrated by comparison with analytical solutions, conventional boundary element, and finite element method.

1. Introduction

The Green's function for the problem of a two-dimensional linear anisotropic elastic solid containing an elliptical hole has been obtained analytically by using Stroh's complex variable formalism [1,2]. Through the use of this Green's function, a special boundary element method (SBEM) for an anisotropic elastic solid containing a single elliptical hole was developed [3] and extended to the vibration analysis [4]. The main feature of SBEM is that no meshes are needed along the hole boundary since the traction-free boundary conditions are satisfied exactly by the Green's function. Thus, a vast of computer time and storage in numerical calculation can be saved. Moreover, due to the exact satisfaction of the hole boundary condition, the results are more accurate than those obtained by the conventional boundary element method (CBEM). Let the minor axis of ellipse be zero, the same fundamental solution can be applied to analyze the problem with a single straight crack [5]. The advantage of no meshes needed on the crack surfaces still works for SBEM. However, due to the restriction of Green's function it is only valid for the straight crack, the interesting cases such as arc-shaped cracks, branched cracks, edge cracks emanating from holes, and cracks under body force discussed in [6] are not included. To solve the general crack problems with the benefit of analytical solutions, several different methods have been proposed in the literature, such as edge function method [7], multi-domain BEM [8–10], displacement discontinuity method [11], dual BEM [12–16], and a special single-domain BEM [6,17].

In addition to the problem of a single hole or crack, there are also many studies discussing the problems of multiple holes and/or cracks. For example, two circular holes in an infinite isotropic medium [18], two elliptical holes in a laminated composite [19], thermal analysis of multiple circular holes [20], multiple cracks [21], and cracks at fastener holes [22]. In order to extend the benefit of SBEM to the problems of multiple holes and cracks, the subregion technique was applied to make each subregion contain only one hole or one crack [23]. The final system of equations for the whole region was then obtained by adding the set of equations for each subregion together with compatibility and equilibrium conditions between their interfaces. Therefore, if several holes and/or cracks appear in the anisotropic elastic solid, the system of equations will become complicated due to the requirement of compatibility and equilibrium along all the interfaces of subregions.

To avoid the trouble caused by the subregion technique, in this study the above-mentioned SBEM is improved by employing the concept of the coupling of boundary element and finite element. Although this concept has been mentioned long time ago [24,25], most of the works divide the body into two domains. One domain is solved by finite element method, and the other is by boundary element

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method. Instead of two totally different domains, in this study the entire domain is divided into N domains (N is the number of holes and cracks) and each domain is discretized according to the rule of SBEM. To assemble all domains smoothly without involving the trouble of subregion technique, SBEM is transformed into an equivalent finite element by using the relation between element nodal force of finite element and surface traction of boundary element. After the transformation, all the elements can be assembled together by following the rule of finite element method [26], and the compatibility and equilibrium considered in the subregion technique will then be satisfied automatically. The method proposed in this paper is therefore called *boundary-based finite element method* (BFEM). Note that the proposed method is different from the other similar name "the scaled boundary finite-element method (sometimes also mentioned in abbreviation by BFEM)" [27], which can be characterized as a fundamental-solution-less BEM solely based on finite element [28]. On the other hand, in our proposed method each region constructed by arbitrary number of boundary nodes is a single finite element transferred from SBEM, in which each element containing a single hole, a single crack, or none of them is enriched by the fundamental solution.

After getting the nodal displacements and nodal forces by the system of equations of finite element, the nodal tractions are calculated from the original system of equations of boundary element. Like the calculation of CBEM, after all the values of tractions and displacements of the boundary nodes are determined, the values of displacements, strains and stresses at any interior can be calculated from the boundary integral equation by setting the free term coefficients to be the Kronecker delta. Unlike the conventional method which usually needs fine meshes along the hole boundary or very fine meshes near the crack tip, in BFEM no meshes are required on the hole/crack surface, and both of the stress concentration factors of hole and the stress intensity factors of crack can be evaluated by using only the remote boundary displacements and tractions. To show the accuracy and efficiency of the proposed BFEM, several numerical examples are executed and compared with the solutions obtained by analytical solutions, CBEM and commercial finite element software ANSYS.

2. A special boundary element for problems of holes and cracks

If body forces are omitted, the boundary integral equations for anisotropic elastostatics can be written as [25]

$$c_{ij}(\boldsymbol{\xi})u_j(\boldsymbol{\xi}) + \int_\Gamma t_{ij}^*(\boldsymbol{\xi},\mathbf{x})\,u_j(\mathbf{x})d\Gamma(\mathbf{x}) = \int_\Gamma u_{ij}^*(\boldsymbol{\xi},\mathbf{x})t_j(\mathbf{x})d\Gamma(\mathbf{x}),\ i,j=1,2,3, \tag{1}$$

where Γ denotes the boundary of the elastic solid; $u_j(\mathbf{x})$ and $t_j(\mathbf{x})$ are the displacements and surface tractions at the field point $\mathbf{x}$ of the boundary Γ; $u_{ij}^*(\boldsymbol{\xi},\mathbf{x})$ and $t_{ij}^*(\boldsymbol{\xi},\mathbf{x})$ are fundamental solutions of displacements and tractions; $c_{ij}(\boldsymbol{\xi})$ is a coefficient dependent on the location of the source point $\boldsymbol{\xi}$, which equals to $\delta_{ij}/2$ for a smooth boundary and $c_{ij}=\delta_{ij}$ for an internal point. The symbol δ_{ij} is the Kronecker delta.

In boundary element formulation, the boundary Γ is approximated by a series of elements, and the points $\mathbf{x}$, displacements $\mathbf{u}$ and tractions $\mathbf{t}$ on the boundary are approximated by the nodal points $\mathbf{x}_n$, nodal displacement $\mathbf{u}_n$ and nodal traction $\mathbf{t}_n$ through different interpolation functions. If the same quadratic variation within each element is assumed for the boundary points $\mathbf{x}$, displacements $\mathbf{u}$ and tractions $\mathbf{t}$, then

$$\begin{aligned}
\mathbf{x} &= \varpi_1\mathbf{x}_m^{(1)} + \varpi_2\mathbf{x}_m^{(2)} + \varpi_3\mathbf{x}_m^{(3)},\\
\mathbf{u} &= \varpi_1\mathbf{u}_m^{(1)} + \varpi_2\mathbf{u}_m^{(2)} + \varpi_3\mathbf{u}_m^{(3)},\\
\mathbf{t} &= \varpi_1\mathbf{t}_m^{(1)} + \varpi_2\mathbf{t}_m^{(2)} + \varpi_3\mathbf{t}_m^{(3)},\ \text{ on the } m\text{th element,}
\end{aligned} \tag{2a}$$

where

$$\varpi_1 = \frac{1}{2}\varsigma(1-\varsigma),\quad \varpi_2 = (1-\varsigma)(1+\varsigma),\quad \varpi_2 = \frac{1}{2}\varsigma(1+\varsigma). \tag{2b}$$

In (2a), a symbol with subscript m and superscript (1), (2) or (3) denotes the value of node 1, 2 or 3 of the *mth* element; the variable ς is the dimensionless coordinate ranging from −1 to 1. Substituting the quadratic approximation assumed in (2) into the boundary integral Eq. (1), and following the standard procedure of boundary element formulation [22], a system of algebraic equations can be written as

$$\sum_{n=1}^{N}\mathbf{Y}_{in}\mathbf{u}_n = \sum_{n=1}^{N}\mathbf{G}_{in}\mathbf{t}_n,\quad i=1,2,\ldots,N, \tag{3}$$

where $\mathbf{Y}_{in}$ and $\mathbf{G}_{in}$ are the matrices of influence coefficients associated with nodes i and n; $\mathbf{u}_n$ and $\mathbf{t}_n$ are the vectors of displacements and tractions at node n; N is the number of nodes.

It's known that if the fundamental solutions for the problems with holes or cracks are employed, their associated boundary elements will possess several advantages than the regular boundary elements, e.g. no meshes are needed along the boundaries of holes or cracks [3,23]. In order to take advantage of this special feature, the fundamental solution derived from the Green's function for the problems of elliptical holes was used in this study, which can be written in matrix form as [2]

$$[u_{ij}^*] = \mathbf{U}^* = 2\mathrm{Re}\{[\mathbf{AF}(\zeta)]^T\},\ [t_{ij}^*] = \mathbf{T}^* = 2\mathrm{Re}\{[\mathbf{BF}_{,s}(\zeta)]^T\}, \tag{4a}$$

where

$$\mathbf{F}(\zeta) = \frac{1}{2\pi i}\left\{<\ln(\zeta_\alpha-\hat{\zeta}_\alpha)>\mathbf{A}^T + \sum_{k=1}^{3}<ln(\zeta_\alpha^{-1}-\bar{\hat{\zeta}}_k)>\mathbf{B}^{-1}\bar{\mathbf{B}}\mathbf{I}_k\bar{\mathbf{A}}^T\right\}, \tag{4b}$$

and $i=\sqrt{-1}$. In (4a,b), $\mathbf{A}$ and $\mathbf{B}$ are material eigenvector matrices; Re denotes the real part of a complex value; superscript T is the transpose of a matrix; the overbar stands for the complex conjugate; the angular bracket < > stands for a 3×3 diagonal matrix in which each component is varied according to its subscript α, e.g., $<z_\alpha> = \mathrm{diag}[z_1, z_2, z_3]$; the symbol $\mathbf{F}_{,s} = \partial\mathbf{F}/\partial s$ where s is the tangential direction; $\mathbf{I}_k$ is the 3×3 diagonal matrix with a unit value at the kk component and all the other components are zero, i.e., $\mathbf{I}_1 = \mathrm{diag}[1,0,0]$, $\mathbf{I}_2 = \mathrm{diag}[0,1,0]$, $\mathbf{I}_2 = \mathrm{diag}[0,0,1]$; the variables $\zeta_\alpha,\alpha=1,2,3$, are related to the complex variables $z_\alpha = x_1+\mu_\alpha x_2$ of the field point $\mathbf{x}=(x_1,x_2)$ by

$$\zeta_\alpha = \frac{z_\alpha + \sqrt{z_\alpha^2 - a^2 - b^2\mu_\alpha^2}}{a - ib\mu_\alpha},\quad \alpha=1,2,3, \tag{5}$$

in which a and b are the major and minor axes of the ellipse respectively, and μ_α, $\alpha=1,2,3$ are the material eigenvalues. Same equation as (5) was also used for the determination of $\hat{\zeta}_\alpha$, which is related to the complex variables $\hat{z}_\alpha = \hat{x}_1 + \mu_\alpha\hat{x}_2$ of the source point $\boldsymbol{\xi} = (\hat{x}_1,\hat{x}_2)$. Note that there will be more than one value of ζ_α (or $\hat{\zeta}_\alpha$) mapped by z_α (or $\hat{z}_\alpha$) through the relation (5). Among them, the one located outside the unit circle of ζ_α (or $\hat{\zeta}_\alpha$), i.e., $|\zeta_\alpha|>1$ (or $|\hat{\zeta}_\alpha|>1$) should be selected [29].

3. Transition of boundary element to finite element

The special boundary element introduced in the previous section is for the anisotropic plates containing only a single hole or crack. If there are multiple holes and cracks inside the body, subregion technique can be applied to make each subregion contains one hole or one crack [23]. The final system of equations for the whole region is then obtained by adding the set of equations for each subregion together with compatibility and equilibrium conditions between their interfaces. Therefore, if several holes and cracks appear in the anisotropic solid, the system of equations will become complicated due to the requirement of compatibility and equilibrium along all the interfaces of subregions. For example, if we have an internal point intersected by four subregions (e.g., point A of Fig. 1), to describe the difference of tractions on