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Comparison between the formulation of the boundary element method that uses fundamental solution dependent of frequency and the direct radial basis boundary element formulation for solution of Helmholtz problems☆

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ABSTRACT

Seeking to validate a solution technique for Helmholtz problems, the Boundary Element Method with Direct Integration, which applies radial basis functions to approach the inertia term, is used to solve numerically problems governed by Helmholtz Equation. The standard Boundary Element formulation that employs the fundamental solution correlated to the Helmholtz Equation and has the frequency of excitation as argument is used for comparison. Thus, examples comprising the direct solution of Helmholtz problems are solved with both Boundary Element formulations and then their results are compared with available analytical solutions.

1. Introduction

The great economic and industrial interest resulting from the exploration of oil deposits, made by modern seismic analysis technologies, produces intense development and improvement of numerical methods in the area of dynamic of structures, seismic data inversion and many others. Many initiatives in this direction can be observed, especially the improvement of numerical processing algorithms coupled with powerful discrete methods, such as the Finite Element Method and the Finite Difference Method. Such initiatives are justified due to the huge computational storage and processing effort required in the treatment of discrete systems with many millions of degrees of freedom. Particularly, this problem is aggravated in the case of 3D dynamic response, since an incremental scheme to advance over time is required.

Due to the unattractive features of a matrix organization, the Boundary Element Method (BEM) has appeared in the background of these applications, which does not mean there is no significant effort in its research context, aiming to improve its mathematical flexibility and to decrease the computational storage effort without losing the quality of its results.

Part of this effort has been made through alternative formulations based on the use of interpolation procedures using radial basis functions [1], since many problems of practical interest are not expressed in terms of differential equations whose operators are self-adjoint or else the inverse integral form is too complicated. The use of these functions also allows the choice of a series of procedures that may, under certain conditions, greatly simplify the processing of the response.

With respect to dynamic problems, there are contents of undeniable interest for the BEM, in which is not demanded huge computational effort, as occurs in step by step time integration procedures. Two important cases consist of analysis of the modal spectrum of the response, which requires as first step calculating eigenvalues and analysis in the frequency domain. These problems are ruled by the Helmholtz Equation. The BEM already has an inverse integral formulation associated with this problem, which uses a fundamental solution which depends on the excitation frequency. Despite its elegance, accuracy and compliance with the mathematical formalism, this formulation presents difficulties related to computational storage effort, and relative lack of flexibility in dealing with problems not concerned with determining directly the response to a known excitement. The solution of a simple eigenvalue problem is one of those limited cases. The matrix related to the system inertia is dependent on the, eigenfrequency, preventing following the classic matrix formulation used for solution of eigenvalue problems.

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In this sense, a first major contribution came with the development of the Dual Reciprocity Boundary Element formulation (DRBEM) [2] that uses radial basis functions as auxiliary tool. The DRBEM allows accessible simulation of transient cases, characteristic value problems, dynamic response problems, and problems characterized by domain sources or actions, which were previously solvable only with costly and relatively complex methods. Although flexible and presenting reasonably results, the DRBEM is exposed to certain numerical inaccuracies, primarily due to matrix conditioning problems deriving from the need of imposing interpolation basis points to represent domain properties, commonly called poles [3].

More recently, it was proposed a new technique based on the use of radial functions, called Direct Radial Basis Interpolation using Boundary Integration (DIBEM) [4].

Unlike DRBEM, the DIBEM formulation proposed here does not require construction of two auxiliary matrices by multiplying classic boundary element matrices **H** and **G** because it directly approximates the complete integral kernel, similar to what is done in an interpolation process, using only one primitive function. Only the transformation of a domain integral into a boundary integral makes DIBEM different from simple interpolation. Therefore, a wide range of different radial functions and a huge number of poles can be used without instability problems, which commonly occur when using the DRBEM.

This work seeks to accurately compare the performance of DIBEM in face of the traditional formulation of the BEM using the fundamental solution depending on the frequency (FSBEM) [5]. There is no expectation that DIBEM results be superior, since in this technique the system inertia is approximated by radial functions. Thus, when dealing with the high frequencies, there is need for better characterization of the system inertia, and under these conditions the approach dictated by DIBEM will certainly provide more inaccurate results.

However, the DIBEM is a technique more versatile than the FSBEM. Its mathematical model allows a wider range of applications using simpler fundamental solution. Unlike the DRBEM, many type of radial basis function may be used without numerical inaccuracies or instabilities. A simple scheme using primitive functions avoids domain integrations, as it will be shown herein.

Anyway, it is important to perform a comparison between the two boundary element formulations concerning the solution of the Helmholtz problems, in which the accuracy is evaluated: this is the main objective of the present study.

2. Helmholtz equation

The Helmholtz equation is one of the most important equations of mathematical physics and engineering. It can be understood as a special case of the generalized scalar field equation in which a potential u (**X**), where **X**=**X** (x_1, x_2), is subjected to a diffusive action generated by the Laplacian operator and reacts proportionally, such as:

$$\nabla^2 u(\mathbf{X}) = -k^2 u(\mathbf{X}) \tag{1}$$

The meaning of the proportionality constant k depends on the physical problem addressed.

One reason for the great importance of the Helmholtz equation is that in dynamic analysis, it can be interpreted as the time harmonic representation of the Acoustic Wave Equation given by:

$$\nabla^2 U(\mathbf{X}, t) = \frac{\rho}{E}\ddot{U}(\mathbf{X}, t) = \frac{1}{c^2}\ddot{U}(\mathbf{X}, t) \tag{2}$$

Where c is the propagation velocity of the acoustic wave, function of the ratio between the bulk modulus and the density [6] and U(X) is the spatial response of the system to any excitations with generalized modal content. In the particular case in which one seeks to determine the response produced by a variable excitation whose frequency ω is known, the potential U(X,t) is given by:

$$U(\mathbf{X}, t) = u(\mathbf{X})e^{-i\omega t} \tag{3}$$

In Eq. (3) i is the imaginary unit and U (X) is the spatial response of the system to the harmonic excitation ω. Thus, the second derivative of U(X,t) with respect to t follows from Eq. (3) to be:

$$\ddot{U}(\mathbf{X}, t) = -\omega^2 u(\mathbf{X})e^{-i\omega t} \tag{4}$$

The application of the Laplacian operator in Eq. (3) and its substitution into Eq. (2) along with Eq. (4), results in the Helmholtz, where k is equal to (ω/c). Furthermore it is clear that the significance of the potential u(**X**) in this case is simple amplitude response, stationary, but which varies from point-to-point in the field considered.

In reality, the problems governed by the Helmholtz equation can be divided into three groups; direct problems, eigenvalue problems and inverse problems.

In the direct problems, the aim is to determine the dynamic equilibrium response u(X) according to a known set of boundary conditions and given value of frequency ω. This is the case discussed in this paper.

In eigenvalue problems, we seek to find all the values of frequencies, called natural frequencies usually denoted by ω_n, capable of generating self equilibrated responses in the system if no external excitations exist. Such frequencies are named natural frequencies or eigenvalues.

In the inverse problem, the system properties are determined through knowledge of applied excitations and the system response.

3. Helmholtz fundamental solution

The Helmholtz fundamental solution is a transfer function that for a given frequency transfers effects of a concentrated force represented by a Dirac delta function from a source point to a field point, the former being the point where the Dirac delta function acts,the latter being the point where the response is measured. Green's function is a denomination also often employed mainly when certain boundary conditions are satisfied. Although contemporary, the study of Green's functions has acquired greater generality and has been encompassed in the context of the Theory of Distributions [7,8].

Therefore, given a two dimensional infinite domain, the fundamental solution u*(**X**,**ξ**,**k=ω/c**) is dependent of frequency ω, and of source ξ and field **X** points location, u* its normal derivative q*(**X**) in η direction, generated at the field point **X** for an arbitrary source point ξ located at an Euclidean distance r of it [5,9], are given by:

$$u^*(\boldsymbol{\xi}; \mathbf{X}) = \frac{1}{2\pi}K_0\left(i\frac{\omega r}{c}\right) \tag{5}$$

$$q^*(\boldsymbol{\xi}; \mathbf{X}) = -\frac{i\omega}{2\pi c}K_1\left(i\frac{\omega r}{c}\right)\frac{\partial r}{\partial n(\mathbf{X})} \tag{6}$$

In Eq. (6), K_0 and K_1 are Bessel functions, respectively of first and second types.

4. BEM with fundamental solution dependent of frequency

In this work, the formulation of the BEM with frequency dependent fundamental solution, named FSBEM, is based on the use of basic mathematical tools of the Theory of Integral Equations. The integral equation associated to the Eq. (1), using as auxiliary function the frequency dependent fundamental solution, Eq. (5), is given as shown below:

$$\int_\Omega \nabla^2 u(\mathbf{X})u^*(\boldsymbol{\xi}; \mathbf{X})d\Omega = -k^2\int_\Omega u(\mathbf{X})u^*(\boldsymbol{\xi}; \mathbf{X})d\Omega \tag{7}$$

Where Ω represents the domain.

The application of integration by parts and the divergence theorem on Eq. (7), operations well documented in literature [10,11], consider-