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A note on the use of the Companion Solution (Dirichlet Green's function) on meshless boundary element methods

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ABSTRACT

Most implementations of meshless BEMs use a circular integration contours (spherical in 3D) embedded into a local interpolation stencil with the so-called Companion Solution (CS) as a kernel, in order to eliminate the contribution of the single layer potential. However, the Dirichlet Green's Function (DGF) is the unique Fundamental Solution that is identically zero at any given close surface and therefore eliminates the single layer potential. One of the main objectives of this work is to show that the CS is nothing else than the DGF for a circle collocated at its origin. The use of the DGF allows the collocation at more than one point, permitting the implementation of a P-adaptive scheme in order to improve the accuracy of the solution without increasing the number of subregions. In our numerical simulations, the boundary conditions are imposed at the interpolation stencils in contact with the problem boundary instead of at the corresponding integration surfaces, permitting always the use of circular integration contours, even in regions near or in contact with the problem domain where the densities of the integrals are reconstructed from the interpolation formulae that already included the problem boundary conditions.

1. Introduction

When dealing with the BEM for large problems, with or without closed form fundamental solution, it is frequently used a domain decomposition technique, in which the original domain is divided into subdomains, and on each of them the full integral representation formulae are applied. At the interfaces of the adjacent subdomains the corresponding full-matching conditions are imposed (local matrix assembly). While the BEM matrices, which arise in the single domain formulation, are fully populated, the subdomain formulation leads to block banded matrix systems with one block for each subdomain and overlaps between blocks when subdomains have a common interface. In the limit of a very large number of subdomains, the resulting internal mesh pattern looks like a finite element grid. The implementation of the subdomain BEM formulation in this limiting case, i.e. a very large number of subdomains, including cells integration at each subdomain has been called by Taigbenu and collaborators as the Green Element Method (GEM) (see [1]). A similar approach based on large number of subdomains but using the Dual Reciprocity Method (DRM) to evaluate the domain integrals at each subdomain, instead of cell integration, has been referred by Popov and Power [2] as the Dual Reciprocity Multi Domain approach (DRM-MD), for more details see Portapila and Power [3].

Meshless formulations of local BEM approaches, see Zhu et al. [4], are attractive and efficient techniques to improve the performance of local BEM schemes. In the meshless BEM the integral representation formulae are applied at local internal integration subregions embedded into interpolation stencils that are heavily overlapped. In this type of approach the continuity of the field variables are satisfied by the interpolation functions avoiding the local connectivity between subdomains or elements needed to enforce the matching conditions between them. Different interpolation schemes can be employed at the interpolation stencils, being the moving least squares shape functions and RBF interpolations the most popular approaches used in the literature. A major advantage of the meshless local BEM formulations in comparison with the classical BEM multi domain decomposition approaches, as the GEM and the DRM-MD, is that the resulting integrands of the integral representation formulae are all regular, instead of singular, since the collocation points are always selected inside the integration subregion.

In the Local Boundary Integral Element Methods (LBEM or LBIEM) the solution domain is covered by a series of small and heavily

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overlapping local interpolation stencils, where a direct interpolation of the field variables is used to approximate the densities of the integral operator, and the boundary conditions of the problem are imposed at the integral representation formula; i.e. at the global system of equations [4–9]. In this type of approach, the domains of integration usually are defined over several stencils, resulting in highly overlapping integration subregions, in addition to the overlapping of interpolation stencils.

As is the case of the interpolation stencils, different shapes of the integration subregion can be considered in the implementation of a meshless BEM, being a circular shape the most popular one (sphere in 3D). As suggested by Zhu et al. [4] (see also Atluri et al. [10], and most of today implementations of meshless BEMs), in the case of a circular integration subregion with a single evaluation point at the centre of the circle, a Companion Solution instead of the Fundamental Solution can be used in the integral representation formula of a given problem in order to eliminate the single layer potential in the integral formulation. However, it is well known in the mathematical literature, that the Dirichlet Green's function is the unique fundamental solution that eliminates the single layer potential from the integral representation formula, whatever the shape of the integration surface is. For clearness in the presentation, we provide here the formal mathematical definition of the different singular solutions considered in this work, i.e. Fundamental solution, Green's function and Dirichlet Green's function. A Fundamental solution of a linear partial differential equation (PDE), or free space Green's function, is a particular solution, i.e. a no unique solution, of the corresponding nonhomogeneous PDE with a Dirac delta function as the nonhomogeneous term, which is singular at the collocation point of the delta function, the Green's function is the unique solution of the same nonhomogeneous PDE, i.e. with a Dirac delta function as the nonhomogeneous term and consequently a singular solution, that satisfies a given homogeneous boundary condition on a prescribed boundary, consequently, the Dirichlet Green's function is the corresponding Green's function satisfying a homogeneous Dirichlet boundary condition. One of the main objectives of this work is to show that the so-called Companion Solution (CS) is nothing more than the Dirichlet Green's function (GF) for a circle collocated at its origin. This should not be regarded as pure semantic meaning of the word (Companion or Green's function), which in the opinion of the authors is important to clarify; since, as shown here, the use of the centre of the circle as the only collocation point of the integral formulation significantly restricts the versatility of the meshless approach.

2. Mathematical formulation and boundary integral represention formula

Let us consider a boundary value problem defined on a two dimensional domain Ω that satisfies a linear partial differential equation (PDE) of the following type:

$$\nabla^2 u(\mathbf{x}) = b\left(\mathbf{x}, u(\mathbf{x}), \frac{\partial u}{\partial x_i}(\mathbf{x})\right), \tag{1}$$

which is written as a non-homogeneous Laplace's equation with non-homogeneous term given by b, and $u(\mathbf{x})$ is the unknown potential field at the point $\mathbf{x} \in \Omega$. The problem definition is completed by specifying the following boundary conditions (BC):

$$u(\mathbf{x}) = u_0(\mathbf{x}) \quad \text{on } \Gamma_1 \tag{2}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = q_0(\mathbf{x}) \quad \text{on } \Gamma_2 \tag{3}$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$, with Γ_1 and Γ_2 are non-intersecting parts of the domain boundary Γ, and the functions u_0 and q_0 are suitably prescribed functions of $\mathbf{x}$.

The integral representation formula for the above linear PDE in terms of the Laplace's fundamental solution is obtained from the Green's second identity in terms of the superposition of surfaces (single and double layers) and volume potentials.

$$c(\xi)u(\xi) = \int_\Gamma q^*(\mathbf{x}, \xi)u(\mathbf{x})d\Gamma_x - \int_\Gamma u^*(\mathbf{x}, \xi)q(\mathbf{x})d\Gamma_x + \int_\Omega bu^*(\mathbf{x}, \xi)d\Omega_x \tag{4}$$

with ξ as the evaluation point, also referred to as collocation point, and $u^*(\mathbf{x}, \xi)$ as the fundamental solution of the Laplace problem, which in the case of two-dimensional problems is given by:

$$u^*(\mathbf{x}, \xi) = \frac{1}{2\pi}\ln\left(\frac{1}{R(\mathbf{x}, \xi)}\right) \tag{5}$$

where $R(\mathbf{x}, \xi)$ is the distance between the integration points $\mathbf{x}$ and collocation point ξ, i.e. $R = |\mathbf{x} - \xi|$, and $q^*(\mathbf{x}, \xi) = \frac{\partial u^*}{\partial n}(\mathbf{x}, \xi)$. The constant value $c(\xi) \in [0, 1]$, being 1 if the point ξ is inside the domain and $\frac{1}{2}$ if the point ξ is on a smooth part of the domain boundary Γ.

In the BEM literature, the approach to obtain the above integral representation formula is sometimes referred to as a weighted residual or reciprocity approach instead of the Green's second identity, which is a misuse of a concept (a weighted residual is an approximate formulation while the Green's identity is an exact representation).

The above integral representation formula is the basis of any meshless BEM approach, where the integration surface Γ and domain Ω are chosen as integration subregions, Γ_i and Ω_i, embedded inside of a corresponding interpolation stencils, which are heavily overlapped. If in the above formulation instead of using the fundamental solution, $u^*(\mathbf{x}, \xi)$, and its normal derivative, $q^*(\mathbf{x}, \xi)$, the Dirichlet Green's function, $G(\mathbf{x}, \xi)$ and its corresponding normal derivative, $Q(\mathbf{x}, \xi)$, are used, follows that Eq. (4) at each integration subregion reduces to:

$$c(\xi)u(\xi) = \int_{\Gamma_i} Q(\mathbf{x}, \xi)u(\mathbf{x})d\Gamma_x + \int_{\Omega_i} bG(\mathbf{x}, \xi)d\Omega_x \tag{6}$$

where by definition over the surfaces Γ_i the value of G is identically zero.

In the case of a circular integration surface Γ_i with radius R_i, the Dirichlet Green's function for a source point, ξ, inside the circle can be obtained from the circle theorem, and given by (for more details see Milne-Thomson [11]):

$$G(\mathbf{x}, \xi) = \frac{1}{4\pi}\ln\left(\frac{R_i^2 R(\mathbf{x}, \xi)^2}{R_0^2 R(\mathbf{x}, \hat{\xi})^2}\right) \tag{7}$$

with the image or reflection point, $\hat{\xi}$, located outside the circle along the same ray of the source point. In the above expression $R(\mathbf{x}, \xi)$ is the distance between the field point $\mathbf{x}$ and the source point ξ given by $R(\mathbf{x}, \xi)^2 = R(\mathbf{x})^2 + R_0^2 - 2R(\mathbf{x})R_0\cos(\theta)$, with R_0 the distance between the source point and the centre of the circle, similarly $R(\mathbf{x}, \hat{\xi})$ is the distance between $\mathbf{x}$ and the image point $\hat{\xi}$, where $R(\mathbf{x}, \hat{\xi})^2 = R(\mathbf{x})^2 + (R_i^4/R_0^2) - 2R(\mathbf{x})(R_i^2/R_0)\cos(\theta)$, with R_i^2/R_0 as the distance from the image point to the centre of the circle and θ as the angle between the vectors $\mathbf{x}$ and ξ from the centre of the circle. When the source point ξ is located at the origin, i.e. $R_0 = 0$, the above expression for the Green's function reduces to:

$$G(\mathbf{x}, \xi) = \frac{1}{2\pi}\ln\left(\frac{R(\mathbf{x})}{R_i}\right) \tag{8}$$

In the meshless BEM literature, the above expression has been referred to as a Companion Solution (see Zhu et al. [4]), but as can be seen from the preceding analysis, the so-called Companion Solution is none other than the Dirichlet Green's function for a circle evaluated at its origin. Sladek et al. [12] appear to recognise this when they mention in their manuscript "it is seen that (8) is the Green's function for the Possion's equation vanishing on the boundary of the circular subregion of radius R_i", without giving any further details. As we will show later this is not just a matter of words meaning, i.e. calling a known function