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A non-singular method of fundamental solutions for two-dimensional steady-state isotropic thermoelasticity problems

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ABSTRACT

We consider a boundary meshless numerical solution for two-dimensional linear static thermoelastic problems. The formulation of the problem is based on the approach of Marin and Karageorghis, where the Laplace equation for the temperature field is solved first, followed by a particular solution of the non-homogenous term in the Navier-Lamé system for the displacement, the solution of the homogenous equilibrium equations, and finally the application of the superposition principle. The solution of the problem is based on the method of fundamental solutions (MFS) with source points on the boundary. This is, by complying with the Dirichlet boundary conditions, achieved by the replacement of the concentrated point sources with distributed sources over the disk around the singularity, and for complying with the Neumann boundary conditions by assuming a balance of the heat fluxes and the forces. The derived non-singular MFS is assessed by a comparison with analytical solutions and the MFS for problems that can include different materials in thermal and mechanical contact. The method is easy to code, accurate, efficient and represents a pioneering attempt to solve thermoelastic problems with a MFS-type method without an artificial boundary. The procedure makes it possible to solve a broad spectra of thermomechanical problems.

1. Introduction

In a large variety of engineering systems, such as nuclear power plants, engines and electronic devices, the simultaneous effects of thermal and mechanical loads on the parts have to be studied. In addition, these loads can also substantially influence natural systems, such as the deforming or cracking of rock, ice, etc. The numerical studies of such systems are, in a large majority of the problems, based on the finite-element method (FEM) [1]. However, there have also been strong developments in mesh-reduction methods in which polygon-like meshes are reduced or avoided [2,3]. A typical example is the boundary-element method (BEM) [4], which is based on a weak formulation with the fundamental solution as a weight function. Another, much simpler, alternative is the method of fundamental solutions (MFS) [5], where the trial functions rely on the fundamental solution and collocation. In the BEM and the MFS, the discretization needs to be performed only on the boundary in the case of the existence of the fundamental solution to the problem. The main advantage of the MFS stems from the fact that only the “pointisation” of the boundary is needed, which completely avoids any of the integral evaluations required in the BEM and makes no principal difference in the numerical implementation between the 2D and 3D cases. In the past decade, the MFS turns out to be applied to an ever-increasing number of different problems, as indicated in the survey papers [6–8]. The MFS was, in conjunction with the method of particular solutions (MPS) and the dual reciprocity method, already applied [9,10] to the numerical solutions of 3D isotropic linear thermoelasticity problems. Later, the 2D linear thermoelasticity [11] problems were also solved by the MFS. A recent application of the MFS for inverse boundary value problems in 2D and 3D static thermoelasticity is given in [12,13]. The MFS requires nodes that are positioned on an artificial boundary located outside the computational domain to avoid the singularity of the fundamental solution and at the same time allow for the collocation of the boundary conditions. The location of the artificial boundary represents the most serious problem of the MFS and presently has to be dealt with either heuristically [14] or by using an optimization procedure [15,16] that requires substantial additional computing time. Young et al. [17,18] made a pioneering proposal to place the source points at the boundary in the MFS. In their approach, the diagonal collocation matrix coefficients were determined directly for simple geometries or by using the results from the BEM, based on the fact that the MFS and the indirect boundary integral formulation are similar in nature. A

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subtracting and add-back technique was used in order to avoid the artificial boundary in [19]. In [20], the diagonal terms are determined by the integration of the fundamental solution on the line segments, formed by using neighbouring points, and the use of a constant solution to determine the diagonal coefficients of the derivatives of the fundamental solution in different coordinate directions. The group of W. Chen [21] improved the evaluation of the singular and nearly singular kernels in the MFS for 2D elasticity problems, based on an evaluation strategy that was originally derived for the BEM [22]. They also [23] proposed a non-singular MFS for determining the diagonal coefficients in the modified MFS by applying a known solution inside the domain, so that the diagonal coefficients from both the fundamental solution and its derivative can be determined indirectly, without using any element or integration concept. Again, this approach is appealing, stable, and accurate, but it is costly for solving large-scale problems due to the need to solve the problem twice. The solution also depends on the choice of the internal reference points. The group also recently proposed the singular boundary method [24,25], where the concept of origin-intensity factors is introduced to circumvent the fictitious boundary. Another very simple desingularised MFS procedure is the method of regularized sources, where the singular fundamental solution behaviour is replaced by the nearly singular behaviour [26,27]. The procedure, however, lacks the accuracy of the derivatives on the boundary. Recently, a new boundary meshless approach called the non-singular MFS (NMFS) was developed by the present authors [28,29] for isotropic and anisotropic elasticity problems based on the boundary distributed source (BDS) method [30]. The NMFS has recently been extended to solve porous media problems with moving boundaries [31] and Stokes flow problems [32], and also developed to solve the multi-body elasticity problems [33] as well as 3D elasticity problems with displacement boundary conditions [34]. The concentrated point sources are, in the NMFS, replaced with the area-distributed and volume-distributed sources in 2D and 3D problems, respectively. They represent an analytical integration of the original singular fundamental solution by preserving the advantage of the diagonal dominance of the system of equations, while they have no troublesome singularity issues. Liu and Šarler [28] used the approach of Šarler [20], which involves the reference solution, to determine the diagonal coefficients of the derivatives of the fundamental solution. The problem with the NMFS is that a careful selection is required for the reference solutions. Recently, Liu and Šarler [35] extended the approach [36] to solve isotropic elasticity problems. In the present paper, the developments [35,36] are extended to isotropic thermoelasticity problems, based on the formulation of Marin and Karageorghis [11]. In this new approach, the diagonal coefficients of the heat fluxes and tractions are determined by assuming the balance of the heat fluxes as well as the forces. Several numerical examples with Dirichlet and mixed boundary conditions are presented. The feasibility and the accuracy of the new approach is demonstrated on a spectrum of solved problems with inclusions and/or voids, involving displacement and traction mechanical boundary conditions, as well as temperature and heat-flux thermal boundary conditions.

2. Governing equations

Consider a 2D isotropic linear thermoelastic solid confined to domain Ω with boundary Γ. Let us introduce a 2D Cartesian coordinate system with orthonormal base vectors $\mathbf{i}_x$ and $\mathbf{i}_y$ and coordinates p_x and p_y of point P with the position vector $\mathbf{p} = p_x\mathbf{i}_x + p_y\mathbf{i}_y$. The governing thermoelastic equations for a 2D steady-state heat conduction and plain-strain thermomechanical equilibrium in an isotropic homogeneous medium are

$$\frac{\partial^2 T}{\partial p_x^2} + \frac{\partial^2 T}{\partial p_y^2} = 0, \tag{1}$$

$$\frac{2\mu(1-\nu)}{(1-2\nu)}\frac{\partial^2 u_x}{\partial p_x^2} + \mu\frac{\partial^2 u_x}{\partial p_y^2} + \frac{\mu}{(1-2\nu)}\frac{\partial^2 u_y}{\partial p_x \partial p_y} - \frac{Eh}{(1-2\nu)}\frac{\partial T}{\partial p_x} = 0,$$
$$\frac{2\mu(1-\nu)}{(1-2\nu)}\frac{\partial^2 u_y}{\partial p_y^2} + \mu\frac{\partial^2 u_y}{\partial p_x^2} + \frac{\mu}{(1-2\nu)}\frac{\partial^2 u_x}{\partial p_x \partial p_y} - \frac{Eh}{(1-2\nu)}\frac{\partial T}{\partial p_y} = 0, \tag{2}$$

where T is the temperature, $\mathbf{u} = u_x\mathbf{i}_x + u_y\mathbf{i}_y$ is the displacement, ν is Poisson's ratio, E is Young's modulus, h is the coefficient of linear thermal expansion, and $\mu = E/2(1+\nu)$. The boundary is divided into two, not necessarily connected, parts $\Gamma^{th} = \Gamma^T + \Gamma^q$ for the thermal, and for the mechanical problem $\Gamma^{me} = \Gamma^u + \Gamma^t$. On the part Γ^T the temperature boundary conditions are given, and on the part Γ^q the heat-flux boundary conditions are given. On the part Γ^u the displacement boundary conditions are given, and on the part Γ^t the traction boundary conditions are given

$$T(\mathbf{p}) = \overline{T}(\mathbf{p}); \quad \mathbf{p} \in \Gamma^T, \qquad q(\mathbf{p}) = \overline{q}(\mathbf{p}); \quad \mathbf{p} \in \Gamma^q, \tag{3}$$

$$u_\varsigma(\mathbf{p}) = \overline{u}_\varsigma(\mathbf{p}); \qquad \mathbf{p} \in \Gamma^u, \quad t_\varsigma(\mathbf{p}) = \overline{t}_\varsigma(\mathbf{p}); \qquad \mathbf{p} \in \Gamma^t, \varsigma = x, y, \tag{4}$$

where $\overline{T}$, $\overline{q}$, $\overline{u}_\varsigma$ and $\overline{t}_\varsigma$ represent known functions. The normal heat flux q on the boundary is related to the temperature gradients by

$$q = -\kappa\left[\frac{\partial T}{\partial p_x}n_x + \frac{\partial T}{\partial p_y}n_y\right], \tag{5}$$

where κ is the thermal conductivity, n_x and n_y denote the coordinates of the outward normal $\mathbf{n}$ at the boundary point $\mathbf{p}$.

In the framework of isotropic linear thermoelasticity, the strain tensor $\boldsymbol{\varepsilon}$ is related to the stress tensor $\boldsymbol{\sigma}$ through the constitutive law of thermoelasticity, i.e.,

$$\varepsilon_{\varsigma\xi} = \frac{1+\nu}{E}\sigma_{\varsigma\xi} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})\delta_{\varsigma\xi} + hT\delta_{\varsigma\xi}, \quad \varsigma, \xi = x, y. \tag{6}$$

Eq. (6) can be expressed in terms of the stresses as

$$\sigma_{\varsigma\xi} = 2\mu\left[\varepsilon_{\varsigma\xi} + \frac{\nu}{1-2\nu}(\varepsilon_{xx} + \varepsilon_{yy})\delta_{\varsigma\xi}\right] - \frac{2\mu(1+\nu)h}{1-2\nu}T\delta_{\varsigma\xi}, \quad \varsigma, \xi = x, y, \tag{7}$$

where the transformation $\nu' = \nu/(1+\nu)$, $E' = E[1-(\nu/(1+\nu))^2]$, $h' = h(1+\nu)/(1+2\nu)$ has to be used for plane-stress problems, and $\delta_{\varsigma\xi}$ is the Kronecker delta

$$\delta_{\varsigma\xi} = \begin{cases} 1, & \varsigma = \xi, \\ 0, & \varsigma \neq \xi, \end{cases} \quad \varsigma, \xi = x, y. \tag{8}$$

The tractions t_x and t_y are defined in terms of the stresses as

$$t_\varsigma = \sigma_{\varsigma x}n_x + \sigma_{\varsigma y}n_y, \quad \varsigma = x, y. \tag{9}$$

3. Solution procedure

3.1. Fundamental solution and particular solution

The fundamental solution T^* of the Laplace heat balance Eq. (1) for the 2D steady-state heat conduction in an isotropic homogeneous medium is

$$T^*(\mathbf{p}, \mathbf{s}) = -\frac{1}{2\pi}\log\frac{r}{r_0}, \tag{10}$$

where $\mathbf{p}(p_x, p_y)$ is a collocation point, $\mathbf{s}(s_x, s_y)$ is a source point, $r = [(p_x - s_x)^2 + (p_y - s_y)^2]^{1/2}$, and r_0 stands for a scaling constant, chosen to ensure that the fundamental solution differs from 0 in the computational domain and at the boundary. The value of the scaling constant should be chosen to be reasonably larger than the maximum distance r of the problem. The corresponding fundamental normal heat flux is (see Appendix A for explicit formula)

$$q^*(\mathbf{p}, \mathbf{s}) = -\kappa\left[\frac{\partial T^*(\mathbf{p}, \mathbf{s})}{\partial p_x}n_x + \frac{\partial T^*(\mathbf{p}, \mathbf{s})}{\partial p_y}n_y\right]. \tag{11}$$