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Regularized boundary integral methods for three-dimensional potential flows

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ABSTRACT

The three-dimensional potential flow problems are solved by a boundary integral equation in this article. The boundary integral equation is regularized by a subtracting and adding-back technique in global elements. This technique utilizes several identities to eliminate the singularities or near singularities of surface integrals. In test cases, the convergence speed of this method for a smooth body is of the order N^{-3} in one direction no matter how high-order quadrature is applied. For nearly singular integrals, several extremely oblate spheroids are tested to verify this method. These results illustrate that this method can effectively improve the nearly singular deficit when it exists. For the non-smooth bodies, the present method is applied to solve the mixed boundary value problems inside two kinds of vessels, which are sloshing motions. At last, some tests are compared between the boundary element methods (local elements) and the present method (global elements).

1. Introduction

The boundary integral equation methods (BIEM) have been widely studied in potential theory, elasticity and acoustics since Jaswon [1], Symm [2], and Hess and Smith [3]. It is classified as a "boundary" method, meaning the computational dimensions can be reduced by one. With the advance of computer technology, the boundary element method (BEM), which emerged in the 1970s, has become the most famous numerical method for solving boundary integral equations. Its early history can be found in the article [4]. Although the boundary element method is popular among scientists and engineers owing to its advantage of saving computer memory, it is somehow tedious to compute the singular integrals during the numerical process. Many textbooks [5–7] have discussed the solving methods for those singular integrals, including polar coordinate transformation, direct limit approach, and analytic regularization. In the traditional BEM, the boundary is discretized into small local elements to approximate its geometry and physical properties. For the sake of using shape functions, it almost inherently generates errors. In order to reduce the numerical error, either the number of elements or the order of shape functions needs to be increased. However, an arbitrary-order shape function is not ready in BEM and a high-order element leads to complex calculations. The quadratic element is almost the highest-order element in practice. Linear and constant elements are much more popular.

In contrast to the traditional BEM, which handles singularities after discretization, the regularization (or called as de-singularized) method deals with the singularities before discretization. Usually, it is more convenient to use regularization methods, especially when dealing with three-dimension problems. To overcome the singular integrals in computation, several techniques were developed. Webster [8] moved the singular point away from the real boundary to avoid the singular integrals. Later, some studies [9,10] found the distance between true boundary and the auxiliary boundary needed to be chosen carefully since it could be sensitive to the solution. Therefore the optimum distance value was determined by a function correlated with the mesh size [11,12]. Such a de-singularized technique is also called as the null-field boundary integral equation method [13–15] for direct method, and named as the method of fundamental solutions (MFS) [16] for the indirect method. On the other hand, instead of moving singular nodes, the "subtracting and adding back" technique smoothens the singularities with some mathematical identities. This approach is different from the de-singularized integral equations in Refs. [8–12]. The quadrature formula can be directly applied on the real boundary, and those quadrature points are taken as collocation points. Therefore, the shape function is unnecessary. If the exact shape of the body is known, the discretization errors in geometry can be completely discarded. The concept of subtracting and adding-back technique can be traced back earlier to Landweber and his co-workers [17,18]. Afterward, Guiggiani developed similar technique to compute the Cauchy principal value

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integrals for elastic problems [19–21]. For two-dimensional cases, Saranen [22] regularized the logarithmic kernel integral equations and theoretically proved its convergence speed for a smooth body with a trapezoidal rule. Afterward the accuracies of both plane exterior and interior problems were discussed in Refs. [23–25]. For three- dimensional cases, the source integral was regularized by utilizing the electrostatic capacitance theorem (also known as equipotential method) and the doublet integral was regularized by the Gauss flux theorem for a closed body [17]. Although the equipotential method is simple, its iteration process needs excessive computing time and sometimes leads to convergence problem when dealing with sharp-shaped boundary. Recently, Hwang [26] gave different formulations to remove the singularities in both source and doublet integrals for an arbitrary element. Meanwhile Zhang et al. [27] also proposed the similar idea to acquire the regularized boundary integral equation, and established the coefficient matrices directly without numerical integration in the same way.

In this article two major objects are focused; one is the problems of flat oblate bodies, and the other is the mixed boundary value problems for non-smooth bodies. First of all, when considering oblate or thin bodies, e.g. airfoils [28], thin-structures [29,30], thin-wall [31] or cracks on plates [32], the nearly singular effect is the major source of numerical errors. Such an integral is more difficult to handle than the singular one. It is not a real singularity in the equation, but it is referred to the numerical drawback from its geometry deficit. When a dipole or source is located near a surface, the value of integrand varies drastically due to the short distance between the load point and its integral region. If the quadrature nodes are not dense enough, comparing with the shortest distance between the load point and the integral surface, the numerical integration will not produce an accurate result. In order to avoid such a situation, one may deploy more quadrature points on the body surface or execute another procedure to smoothen the near singularity. Many researchers have presented various treatments, such as analytical integral method [33], semi-analytical method [34,35], subdivision method [36,37], coordinate transformation method [38] and nonlinear transformations [39–45] to improve the accuracy of nearly singular integrals. As a result, the nonlinear transformations became more popular owing to fewer limitations than other methods. Including the rational transformation [46], the exponential transformation [47], and the sinh transformation [48–51], they all handle the near singularities before applying the numerical quadrature. On the basis of the subtracting and adding back concept [26], Hwang derived auxiliary functions by use of Stokes' theorem to alleviate the near singularity for both doublet and source integrals in three- dimensional problems [26,28]. Although they can be applied to all kinds of boundary with any quadrature formula, the accuracy of this method has not been well understood. The present study applies these formulas to solve the potential flows of oblate bodies and analyzes the efficiency and accuracy of this method. Those solutions are compared with the equipotential method [17] and the traditional BEM. Secondly, this article also investigates the mixed boundary value problems for non-smooth bodies, for example, the sloshing behavior inside a liquid container. Sloshing phenomenon is very important in design of water, chemicals or petroleum tanks. Many useful numerical schemes, such as finite difference method (FDM) [52], finite element method (FEM) [53], volume of fluid (VOF) method [54] and boundary element method (BEM) [55] have been developed for solving such a problem. The physical quantities, such as the velocity of fluid, are calculated at every time step to catch the free-surface profile. The regularized boundary integral method can solve such problems with superior efficiency since only those nodes on the boundary are actually computed. Two kinds of vessels are presented in this article. Detailed discussions are made in the following sections.

2. Three-dimensional boundary integral formulae

The following boundary integral equation is established for solving Laplace's equation. The fundamental solution of Laplace's equation in three dimensions can be expressed as:

$$G=\frac{-1}{4\pi r(p,\,q)}, \tag{1}$$

where $r(p, q)$ is the distance between the load point p and the field point q which locates on the boundary S. Assuming the velocity potential ϕ satisfies Laplace's equation; insert Green's function and the velocity potential into Green's second identity. The potential ϕ can be expressed in an integral form

$$c(p)\phi(p)+\int_S \phi(q)\frac{\partial G}{\partial n}dS=\int_S G\frac{\partial \phi(q)}{\partial n}dS, \tag{2}$$

where $\boldsymbol{n}$ denotes the unit outward normal vector on the boundary, and $c(p)=\int_S \frac{\partial G}{\partial n}dS$ is a coefficient depending on the location of point p. Physically, this coefficient can be interpreted as the flux on the boundary due to a unit source at p. Therefore $c(p)$ is 1 when p is inside the domain, 1/2 when p is on the smooth part of boundary, and 0 when p is outside the domain.

3. Regularized boundary integral methods

The integrals in Eq. (2) involve singularities in both surface integrals when p and q coincide. Fortunately, the normal derivative of source function is quite easy to eliminate. The alternative form of Eq. (2) can be rearranged as follows

$$\int_S [\phi(q)-\phi(p)]\frac{\partial G}{\partial n}dS-\int_S G\frac{\partial \phi(q)}{\partial n}dS=0. \tag{3}$$

The integrand of the first integral in Eq. (3) becomes bounded and the value can be treated as zero in an average sense when p and q coincide on a smooth region. However, the other singularity in source integral needs much more work. To remove the singularity in the source integral, a procedure was proposed by Landweber and Macagno [17]. They apply a source distribution on the boundary to form an equipotential surface on it. The source distribution satisfies a homogeneous integral equation [1].

$$\int_S e(q)\frac{\partial G}{\partial n_p}dS=-\frac{1}{2}e(p), \tag{4}$$

where $e(p)$ is the function of source distribution, and n_p is the unit outward normal vector of point p. Since the numerical values of kernel function in Eq. (4) can be directly obtained from the dipole term in Eq. (3), an economical way to solve this distribution is to use an iterative process, such as:

$$e_{n+1}(p)=e_n(p)-2\int_S [e_n(q)\frac{\partial G}{\partial n_p}-e_n(p)\frac{\partial G}{\partial n}]dS \tag{5}$$

The equipotential value ϕ_e can be evaluated in the following form

$$\phi_e=-\int_S e(q)GdS, \tag{6}$$

and this value is a constant including the boundary and its inside. Then, the subtracting and adding-back technique is applied to remove the source singularity in the last integral of Eq. (2) such as:

$$\int_S G\frac{\partial \phi(q)}{\partial n}dS=\int_S G[\frac{\partial \phi(q)}{\partial n}-\frac{e(q)}{e(p)}\frac{\partial \phi(p)}{\partial n}]dS+\frac{\phi_e}{e(p)}\frac{\partial \phi(p)}{\partial n}. \tag{7}$$

4. Boundary integrals on partial surface of body

In the last section, the surface integral deals with the whole body surface, but these formulas may not always work. For example, when