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Analysis of the inherent instability of the interpolating moving least squares method when using improper polynomial bases

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ABSTRACT

This paper first discusses the inherent instability of the interpolating moving least squares (IMLS) method. In the original IMLS method, non-scaled polynomial bases are used. Theoretical and numerical results indicate that the stability of the original IMLS method decreases as the separation distance decreases. Then, using shifted and scaled polynomial bases, a stabilized algorithm of the IMLS method is proposed and analyzed. As an application, the stabilized IMLS method is finally introduced into the meshless Galerkin boundary node method (GBNM) to produce a stabilized GBNM for potential problems and Stokes problems. Numerical examples are given to demonstrate the stability and convergence of the presented stabilized algorithms.

1. Introduction

The moving least squares (MLS) method [1] is an approximation method to construct continuous functions from a set of point values based on the computation of a weighted least squares approximation. Because the numerical approximation starts from scattered nodes instead of elements, many meshless methods, such as the meshless local Petrov–Galerkin method [2], the element-free Galerkin (EFG) method [3,4], the boundary node method (BNM) [5,6] and the symmetric Galerkin BNM [7,8] have been developed using the MLS method. In these meshless methods, the MLS method is used to form shape functions. In recent years, some MLS variants [9], such as the improved MLS [10,11], the complex variable MLS [12] and the improved interpolating MLS [13,14] have also been presented.

A disadvantage of the MLS method is that its shape function lacks the property of Kronecker delta function. Thus, boundary conditions in MLS-based meshless methods cannot be implemented directly and easily. Many specific techniques have been developed to implement boundary conditions [15]. To restore the delta function property of the MLS method, Lancaster and Salkauskas [1] further developed an interpolating moving least squares (IMLS) method. The IMLS method is derived based on the MLS method by using singular weight functions. In the IMLS method, the shape function possesses the delta function property, so implementing boundary conditions in IMLS-based meshless methods is much easier than that in MLS-based meshless methods.

By using the IMLS method to form meshless shape functions, Kaljevic and Saigal [16] proposed an improved EFG. Besides, by revising the formulae of the IMLS method, Ren et al. [17,18] obtained new simpler formulae of the IMLS method, and then proposed the interpolating boundary element-free method and the interpolating EFG method. Moreover, Maisuradze et al. [19] used the IMLS method to fit potential energy surfaces in one dimensional chemical application, while Netuzhylov [20] used the meshless collocation method based on the IMLS method to solve boundary value problems. In these IMLS-based meshless methods, boundary conditions are applied directly and easily, and the number of unknown coefficients in the trial function of the IMLS method is less than that in the trial function of the MLS method. Therefore, these IMLS-based meshless methods have high computational efficiency and precision [21–23].

As in the MLS approximation, continuous functions in the IMLS method are constructed from a set of point values by computing a weighted least squares approximation. Therefore, a drawback of the IMLS method is that the moment matrix may become ill-conditioned or singular. This makes that the coefficients in the IMLS method and thus the IMLS shape functions are prone to round-off errors. In fact, it has been noted from our numerical experiments that if the separation distance is chosen to be too small, or if the number of nodes is chosen to be too large, the accuracy of the IMLS interpolation deteriorates. Besides, the

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associated interpolation error tends to grow with the decrease of the separation distance, indicating a state of instability. The main aim of this paper is to show and solve the inherent instability of the IMLS method.

In Refs. [24,25], Mirzaei et al. developed a shifted and scaled polynomial basis function to stabilize the MLS approximation. In Ref. [24], the relationship between the condition numbers and the determinants of the moment matrix in the MLS and those in the stabilized MLS is presented and proved. Besides, in Ref. [25] it was proved that, using the new basis, the minimum eigenvalue of the corresponding moment matrix is bounded independent of the fill distance, which means that the stabilized MLS approximation is theoretically stable.

In this paper, the inherent instability of the IMLS method is studied theoretically and verified numerically. By estimating the determinant and the condition number of the moment matrix in the IMLS method, we find that the stability of the IMLS method deteriorates severely as the separation distance decreases. Since the usually used polynomial basis functions in the IMLS method are non-scaled, the analyzed results indicate that the inherent instability of the IMLS method may be caused by using improper polynomial bases. Then, with the aid of the shifted and scaled polynomial basis function [24,25], a stabilized IMLS method is developed. Theoretical analysis shows that both the determinant and the condition number of the moment matrix in the stabilized IMLS method are invariable with respect to the separation distance. Thus, the stabilized IMLS method prevents the instability occurrence.

The Galerkin boundary node method (GBNM) [7,8] is a symmetric and boundary-only meshless method that combines a variational form of boundary integral equations (BIEs) for governing partial differential equations with the MLS approximation for generation of the trial and test functions. Compared with other MLS-based boundary type meshless methods, the GBNM keeps the symmetry and positive definiteness of the variational formulation in the process of numerical implementation. Numerical applications and theoretical error estimates of the GBNM have been presented for problems in potential theory [7,8], linear elasticity [26] and fluid mechanics [27–29].

In this paper, the stabilized IMLS method is further introduced into the GBNM to produce a stabilized GBNM. Details of numerical implementation of the stabilized GBNM are presented for potential problems and Stokes problems. Unlike the GBNM, the stabilized GBNM is a direct numerical method in which the basic unknown quantity is the real solution of nodal variables. Besides, boundary conditions in the stabilized GBNM can be applied directly and easily. Moreover, the stabilized GBNM is expected to have better computational stability and convergence.

The rest of this paper is organized as follows. Section 2 gives the IMLS method and its stability analysis. In Section 3, a stabilized IMLS method is developed and analyzed. Then, by using the stabilized IMLS method, a stabilized GBNM is presented for potential problems and Stokes problems in Section 4. Finally, numerical examples and conclusions are provided in Sections 5 and 6, respectively.

2. The IMLS method and its stability analysis

2.1. Notations

Let $X=\{\mathbf{x}_i\}_{i=1}^N$ be a set of N nodes in a bounded domain $\Omega\subset\mathbb{R}^n$, where $n=1, 2, \ldots$ denotes the dimension of the domain. A generic point in $\mathbb{R}^n$ is denoted as $\mathbf{x}=(x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ or $y=\left(y_1, y_2, \ldots, y_n\right)^{\mathrm{T}}$.

For the set of nodes $X=\{\mathbf{x}_i\}_{i=1}^N$, the fill distance is defined as [24,25,30]

$$h_{X,\Omega}=\sup_{\mathbf{x}\in\Omega}\min_{1\le j\le N}\left\|\mathbf{x}-\mathbf{x}_j\right\|_2$$

and the separation distance is defined as

$$q_X=\frac{1}{2}\min_{i\neq j}\left\|\mathbf{x}_i-\mathbf{x}_j\right\|_2$$

A set X of data sites is said to be quasi-uniform with respect to a positive constant C if

$$q_X\le h_{X,\Omega}\le Cq_X \tag{1}$$

For any $\mathbf{x}\in\Omega$, assume that the influence domain of $\mathbf{x}$ is

$$\Re\left(\mathbf{x}\right)\triangleq B\left(\mathbf{x},\rho\right)=\left\{\mathbf{y}\in\mathbb{R}^n\colon\ \|\mathbf{x}-\mathbf{y}\|_2<\rho\left(\mathbf{x}\right)\right\}$$

where $B\left(\mathbf{x},\rho\right)$ stands for the ball of radius $\rho(\mathbf{x})$ centered at $\mathbf{x}$.

In particular, the influence domain of the node $\mathbf{x}_i$ is

$$\Re_i\triangleq\Re\left(\mathbf{x}_i\right)=\left\{\mathbf{y}\in\mathbb{R}^n\colon\ \left\|\mathbf{x}_i-\mathbf{y}\right\|_2<\rho_i\right\}$$

where ρ_i is the radius of $\Re_i$. As in Refs. [24,25,30], ρ_i can be chosen to be proportional to the fill distance $h_{X,\Omega}$.

The set $\{\mathbf{x}_i\}_{i=1}^N$ is used to define a finite open covering $\{\Re_i\}_{i=1}^N$ of Ω composed of N balls $\Re_i$ centered at $\mathbf{x}_i$. Moreover, given $\mathbf{x}\in\Omega$, assume that there are $\tau(\mathbf{x})$ nodes $\mathbf{x}_i$ such that $\mathbf{x}\in\Re_i$. Clearly, $\tau(\mathbf{x})$ is not a fixed number and varies with the point $\mathbf{x}$. Then, we use the notation $I_1, I_2, \ldots, I_{\tau(\mathbf{x})}$ to express the global sequence numbers of these nodes, and define

$$\wedge\left(\mathbf{x}\right)=\left\{I_1, I_2, \ldots, I_{\tau(\mathbf{x})}\right\}$$

Let

$$w_i(\mathbf{x})=\begin{cases}\varphi\left(\dfrac{\left\|\mathbf{x}-\mathbf{x}_i\right\|_2}{\rho_i}\right)\left\|\dfrac{\mathbf{x}-\mathbf{x}_i}{\rho_i}\right\|_2^{-\alpha}, & \mathbf{x}\in\Re_i,\quad i=1, 2, \ldots, N\\ 0, & \mathbf{x}\notin\Re_i,\end{cases} \tag{2}$$

be the weight function used in the IMLS method, where the function φ is nonnegative, compactly supported in the unit ball $B(0, 1)$, γ-th times continuously differentiable, and its derivatives up to order γ are bounded. Generally, the function φ can be chosen to be the constant one or any weight function used in the MLS approximation.

Let

$$\mathbf{p}(\mathbf{x})=\left[p_0\left(\mathbf{x}\right), p_1\left(\mathbf{x}\right), \ldots, p_m\left(\mathbf{x}\right)\right]^{\mathrm{T}},\quad \mathbf{x}\in\Omega \tag{3}$$

be a basis for the space $\mathbb{P}_{\hat{m}}^n$, where $\mathbb{P}_{\hat{m}}^n$ denotes the space of n-variate polynomials of degree $\hat{m}$ of dimension $\frac{(\hat{m}+n)!}{\hat{m}!\,n!}:=m+1$. For any $j=0, 1, 2, \ldots, m$, the largest degree of $p_j\left(\mathbf{x}\right)$ is denoted by $\hat{j}$. Here, $\hat{j}$ depends only on j and the spatial dimension n.

As in the MLS approximation, the commonly used basis functions in the IMLS method are non-scaled. For example, in 1D space $(n=1)$, $\hat{j}=j-1$ for all $j=0, 1, 2, \ldots, m$, and a complete basis of degree $\hat{m}$ is given by

$$\mathbf{p}(x)=\left[1, x, x^2, \ldots, x^{\hat{m}}\right]^{\mathrm{T}},\quad x\in\mathbb{R}$$

In 2D space $(n=2)$, we have $\hat{0}=0$, $\hat{1}=\hat{2}=1$, $\hat{3}=\hat{4}=\hat{5}=2$, $\hat{6}=\hat{7}=\hat{8}=\hat{9}=3$, and a complete cubic basis is given by

$$\mathbf{p}(\mathbf{x})=\left[1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1x_2^2, x_1^2x_2, x_2^3\right]^{\mathrm{T}},$$
$$\mathbf{x}=(x_1, x_2)^{\mathrm{T}}\in\mathbb{R}^2,\quad \hat{m}=3$$