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A generalized beta finite element method with coupled smoothing techniques for solid mechanics

W. Zeng [a,b,*], G.R. Liu [b], C. Jiang [c,b], T. Nguyen-Thoi [d,e], Y. Jiang [f]

[a] CEAS-Biomedical Engineering (BME), University of Cincinnati, 2901 Woodside Dr., Cincinnati, OH 45221, USA
[b] CEAS-School of Aerospace Systems, University of Cincinnati, 2851 Woodside Dr., Cincinnati, OH 45221, USA
[c] State Key Laboratory of Advanced Technology of Design and Manufacturing for Vehicle Body, Hunan University, 410082, China
[d] Division of Computational Mathematics and Engineering, Institute for Computational Science (INCOS), Ton Duc Thang University, Ho Chi Minh City, Viet Nam
[e] Faculty of Civil Engineering, Ton Duc Thang University, Ho Chi Minh City, Viet Nam
[f] Hefei General Machinery Research Institute, 888 West Changjiang Rd., Hefei 230031, China

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ABSTRACT

This paper presents a generalized smoothing techniques based beta finite element method (βFEM) to improve the performance of standard FEM and the existing smoothed finite element methods (S-FEM) in solid mechanics. As we know, the edge-based (for 2D) or face-based (for 3D) strain smoothing techniques can bring much more accurate solutions than standard FEM, and offer lower bounds for force driven problems. The node-based smoothing technique with “overly-soft” feature, on the other hand has a unique property of producing upper bound solutions. This work proposes a novel generalized S-FEM with the smoothing domains generated based on both edges/faces and nodes. An adjustable parameter β is introduced to control the ratio of the area of edge/face-based and node-based smoothing domains. It is found that nearly exact solutions in strain energy can be obtained by tuning the parameter, making use of the important property that the exact solution is bonded by the solutions of NS-FEM and ES/FS-FEM. Standard patch tests are likewise satisfied. A number of numerical examples (static, dynamic, linear and nonlinear) have shown that the present βFEM method is found to be ultra-accurate, insensitive to mesh quality, temporal stable, capable of modeling complex geometry, immune from volumetric locking, etc.

1. Introduction

The standard constant finite elements such as 3-node triangular or 4-node tetrahedral elements (T-elements) were popular and preferred in practical mechanics problems for many years, as they offer many advantages such as convenience in FE implementation, high mesh quality, adaptive analysis with mesh rezoning, etc. And sometimes triangular/ tetrahedral mesh (T-mesh) may be the only option for mesh generation of complex geometries (e.g., biomechanical problems with irregular geometrical shapes). However, compared with quadrilateral/hexahedral meshes, the T-mesh using constant strain T-elements has its own numerical drawbacks including the inaccuracy, shear and volumetric locking due to excessive stiffness, especially for large deformation problems. As such, it is usually not recommended to use T-mesh in commercial FEM software packages.

In order to overcome the volumetric locking for plane strain problems and poor accuracy in stress solution, some new FEM approaches have been developed including supplementing the element displacement field with additional nodes and utilizing reduced numerical integration rules to calculate the element stiffness matrix. However, these procedures are not applicable or compatible with constant strain T-elements. T-mesh with second-order or higher-order elements is thought to a good option to avoid the locking issues, but it may be ineffective for extremely large deformation problems due to the intermediate nodes [1,2]. In order to dealing with these element defects of T-mesh, a number of researchers made their efforts to improve it in the past 30 years. For example, Allman [3,4] improved the accuracy of triangle elements by using vertex connectors which included rotations. However, it exhibited an unusual type of zero energy mode, in addition to the rigid body movements. Reference [5] made a critical assessment of the Allman's triangular membrane element with drilling degrees of freedom via examining the performance of the element combined with a triangular plate bending element. Huang et al. [6] modified Allman's triangular planar element with drilling degree of freedom and dealt with spurious energy mode by an introduced constraint which ensures that the drilling degree

* Corresponding author at: CEAS-Biomedical Engineering (BME), University of Cincinnati, 2901 Woodside Dr., Cincinnati, OH 45221, USA.
E-mail addresses: zengwe@mail.uc.edu, zeng.work@gmail.com (W. Zeng).

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of freedom is a true rotation in elasticity. Piltner and Taylor [7] developed the enhanced triangle elements to deal with nearly incompressible plane strain problems. However, the requirement of more degrees of freedom has limited the practical applications of these methods. In References [8,9], elements with rotational freedom were also designed to improve the bending performance or stiffness matrices for planar triangular elements. Reference [10] proposed a weighted least-squares formulation for deriving constant strain T-elements, which claimed to be possible to eliminate volumetric locking. In Ref. [11], it introduced a node-based uniform strain element for T-mesh and it is capable in avoiding the volumetric locking and reducing the effects of shear locking for static linear elastic problems. Reference [12] used bubble function displacements in conjunction with the assumed strain formulation to construct triangular solid shell elements for precluding membrane locking effect.

In the past several years, Liu and his group developed smoothed finite element methods (S-FEM) [13–19] by introducing the gradient/strain smoothing techniques to FEM settings and using direct (no mapping) point interpolation for computing shape functions. The gradient/strain smoothing techniques using Green's theorem have been exploited in the past few decades for the quasi-conforming elements for plates and shells [20], stabilizing nodal integration of meshfree methods [21,22] and natural element method [23]. The essential idea of S-FEM is to utilize a standard first-order finite element mesh (in particular T-mesh) to build numerical models with good performance [24]. In S-FEM, the compatible strain field is constructed in a Galerkin weak form model to produce some good properties. Compared with the element-based implementation in the standard FEM, the S-FEM models evaluate the weak form based on smoothing domains. The smoothing domains can be constructed within the elements but usually beyond the elements, which is able to bring in the information of the neighboring elements. According to different fashions in the creation of smoothing domains, several different types of S-FEM models have been proposed: the cell-based smoothed FEM (CS-FEM) [16,19], node-based smoothed FEM (NS-FEM) [18], edge based smoothed FEM (ES-FEM) [15,17], face-based smoothed FEM (FS-FEM) [25], etc. Compared with the standard FEM, the overestimation behavior of stiffness values shall be reduced or alleviated in S-FEM and it significantly improves the accuracy of both primal and dual quantities [26]. In addition, the evaluation of shape function derivatives involved in FEM is avoided in S-FEM. The applications of S-FEM models in elasticity have shown they are insensitive to mesh distortion (compared with standard FEM) due to the absence of isoparametric mapping [27,28]. Furthermore, an S-FEM model can use the same background mesh as the standard FEM model, which does not require introducing additional degrees of freedom.

The intensive numerical studies have already demonstrated that the class of S-FEM models shows some advantages over standard FEM [24]. Among these S-FEM models, the ES-FEM (or FS-FEM for 3D) possesses some properties such as: i) ES/FS-FEM can produce solution with properties of super-convergence and higher accuracy compared with corresponding FEM model; ii) it usually generates lower bound to the exact solution in text of strain energy, but still has the feature of overestimation of stiffness; iii) it can use T-mesh which can be conveniently generated especially for complex geometries; iv) the ES/FS-FEM models are always stiffer than NS-FEM or FEM, partially due to the number of edges is always larger than the number of nodes with the same background mesh; v) the vibration models using ES/FS-FEM are often temporally stable and there are no spurious non-zeros energy modes found in free vibration analysis [24]. Meanwhile, the NS-FEM has some interesting properties [29–31]: i) it has the unique upper bound property in strain energy as it may extremely soften the over-stiffness of the corresponding standard FEM model; ii) it achieves accurate and often super-convergent stress solutions; iii) it is effective in overcoming volumetric locking; iv) it works effectively with T-mesh; v) it performs spatially stable but may be temporally instable with non-zero-energy spurious modes.

Considering the fact that the ES-FEM is capable of producing the accurate solution from the lower bound (better than standard FEM) and the NS-FEM can approximate the solution from the upper bound, a generalized mixed smoothed FEM model can be naturally conceived in order to obtain the exact or close-to-exact solution measured in an energy norm. Another fascinating aspect is that the generalized smoothed FEM can be versatile and may inherit the merits from both the NS-FEM and ES/FS-FEM. In this work, a novel ultra-accurate generalized smoothing techniques based beta finite element method (βFEM) based on T-mesh is developed and then applied in different mechanics problems, particularly for 3D solid mechanics problems. In βFEM, the smoothing domains will be constructed by a mixed fashion of node-based and edge/face-based smoothing techniques, in which the adjustable parameter $\beta \in [0, 1]$ tunes the portion of area of the node-based and edge-based smoothing domains. The idea of βFEM can be regarded as a utilization of the overestimation property of ES/FS-FEM and the unique under-estimation property of NS-FEM using T-elements, and hence it can be "tuned" to have good features of both methods. Since both the NS-FEM and ES/FS-FEM with T-elements are spatially stable [24], the presented βFEM will be spatial stable and the convergence can be guaranteed. In addition, the scheme ensures the variational consistence and the compatibility of the displacement field, which ensures reproducing linear field exactly [32–34].

The paper aims to propose and formulate the novel generalized βFEM for solid mechanics problems with first-order triangular/tetrahedral mesh, using the mixed edge /face-based and node-based strain smoothing techniques. The governing equation and different smoothing techniques utilized in this work will be briefly introduced in Section 2. The idea of βFEM for both 2D and 3D problems will be presented in Section 3. Section 4 considers the implementation aspects for vibration analysis and large deformation problems which will be shown in subsequent numerical examples. The standard patch test and numerical examples will be discussed in Sections 5 and 6. The conclusion will be summarized in the last section.

2. Background of the problem and strain smoothing techniques

The target of our method is to solve the solid mechanics problems using the weakened weak (W^2) Galerkin formulation [33]. For example, we can consider an elastic deformable body occupying domain Ω, subjected to the body force $\mathbf{f}^b$ and the traction $\mathbf{f}^t$ on the natural boundary Γ_t. The object undergoes arbitrary virtual displacements with the compatible virtual strains $\delta\boldsymbol{\varepsilon}$ and internal displacement $\delta\mathbf{u}$. The dynamic equilibrium equations, which contain the inertial and damping forces, can be described in the following form:

$$\int_{\Omega} \delta\boldsymbol{\varepsilon}^{\mathrm{T}} D \boldsymbol{\varepsilon} \mathrm{d}\Omega - \int_{\Omega} \delta\mathbf{u}^{\mathrm{T}} [\mathbf{b} - \rho\ddot{\mathbf{u}} - c\dot{\mathbf{u}}] \mathrm{d}\Omega - \left(\int_{\Omega} \delta\mathbf{u}^{\mathrm{T}} \mathbf{f}^{b} \mathrm{d}\Omega + \int_{\Gamma_t} \delta\mathbf{u}^{\mathrm{T}} \mathbf{f}^{t} \mathrm{d}\Gamma \right) = 0 \tag{1}$$

where $\mathbf{D}$ is the Hooke matrix of elastic constants which is related to modulus E and Poisson's ratio ν. For static problem, the second term in Eq. (1) will be vanished. The strain tensor $\boldsymbol{\varepsilon}$ can be expressed by displacement $\mathbf{u}$ using compatibility relation:

$$\boldsymbol{\varepsilon} = \nabla_S \mathbf{u}(\mathbf{x}) \tag{2}$$